Discrete Optimization 2023 (EPFL): Problem set of week 6

March 30, 2023

1. Let A be an $m \times n$ matrix with rows $a_1, \ldots, a_m$ and let $b \in \mathbb{R}^m$ be given. Consider the polyhedron P defined by $A\vec{x} \leq \vec{b}$.

   Assume that $I = \{1, 2, \ldots, n\}$ is a basis, but not a feasible basis. Denote by Q the point that is the intersection of the n hyperplanes $\{\langle a_i, x \rangle = b_i\}$ for $i = 1, \ldots, n$.

   Prove that for every $\lambda_1, \ldots, \lambda_n > 0$ there is α such that the hyperplane $H = \{\langle \sum_{i=1}^{n} \lambda_i a_i, x \rangle = \alpha$ separates Q and P.

2. Let P be a (bounded) polytope in $\mathbb{R}^3$ with vertices $v_1, \ldots, v_k$. Let $\vec{c} \in \mathbb{R}^3$ be such that $\langle c, v_i \rangle \neq \langle c, v_j \rangle$ for every $i \neq j$. Assume that P is a *simple* polytope, in the sense that every vertex has precisely 3 neighbors.

   We say that a vertex v of P is of type 1 if precisely two of its three neighbors have their scalar product with c larger than the scalar product of v and c. We say that v is of type 2 if precisely two of its three neighbors have their scalar product with c smaller than the scalar product of v with c.

   Show that the number of vertices of type 1 is always equal to the number of vertices of type 2. Conclude that every simple polytope in $\mathbb{R}^3$ must have an even number of vertices.

   Is it true also in dimensions $2, 4, 5, 6, 7$?

   hint: observe that a vertex of type 1 is the "lowest" vertex of precisely one face of dimension 2 of P. A vertex of type 2 is the "highest" vertex of precisely one face of P of dimension 2.

Recall that K is a convex set if for every $x, y \in K$ and every $0 \leq \lambda \leq 1$ we have $\lambda x + (1-\lambda)y \in K$.

3. Let P be a set of vectors (points) in $\mathbb{R}^n$. The *cone* generated by P is the set of all finite sums $\sum a_i p_i$ such that $a_i \geq 0$ for every i and $p_i \in P$. Show that the cone generated by a set of points is always a convex set.

4. Let K be a convex set and assume $p_1, \ldots, p_t \in K$. Show that $\sum \lambda_i p_i \in K$ for every positive $\lambda_1, \ldots, \lambda_t$ whose sum is equal to 1,

   Hint: use induction on t. For $t = 2$ it is just the definition of being convex.

5. Given a set P of points, the *convex hull* of P is the set of all finite sums of the form $\sum \lambda_i p_i$ where $\sum \lambda_i = 1$ and the λ_i's are all nonnegative and the p_i's are points in P. Show that the convex hull of a set P of points is a convex set. Conclude it is the smallest convex set containing P.