Discrete Optimization 2023 (EPFL): Problem set of week 5

March 23, 2023

1. Consider the simplex (tetrahedron) P in $\mathbb{R}^3$ whose vertices are $(1,0,0)$, $(-1,1,0)$, $(-1,-1,0)$, and $(0,0,1)$. Find all the vectors $\overrightarrow{c}$ such that the maximum of $\langle \overrightarrow{c}, \overrightarrow{x} \rangle$ on P is at the vertex $(0,0,1)$.

2. Let S be the set of all 2^n vectors with coordinates that are equal either to $+1$ or -1. Let P be the set of all linear combinations of the vectors in S with coefficients greater than or equal to 0 and smaller than or equal to 1. Show that P is convex and find at least 3 distinct vertices of P.

3. Let P be the tetrahedron whose vertices are $(1,2,3), (2,1,-1), (1,1,0)$, and $(2,1,-3)$. Find a hyperplane H that supports P and intersects P at the edge with vertices $(1,2,3)$ and $(1,1,0)$.

4. Let $\overrightarrow{a}_1, \ldots, \overrightarrow{a}_{n+1}$ be $n+1$ vectors in $\mathbb{R}^n$ such that every n of them are linearly independent. Let $\overrightarrow{b} \in \mathbb{R}^{n+1}$ be a vector with positive coordinates. Let A be the matrix whose rows are $a_1, \ldots, a_{n+1}$. Show that if $\sum_{i=1}^{n+1} \overrightarrow{a}_i = 0$, then the polyhedron $P = \{x \mid Ax \leq b\}$ is bounded and not empty.

5. (Extra practising exercise – check your solutions with the computer)

   Find all the vectors of P that is defined by the following system of inequalities.

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 10 \end{pmatrix} \overrightarrow{x} \leq \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ \frac{1}{3} \end{pmatrix}$$