Discrete Optimization 2023 (EPFL): Problem set of week 4

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1. Let A be the matrix
$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}.$$
Let $\vec{b} = (1, 1, 1, 1, 1)$ and let $P = \{\vec{v} = (x, y, z) \in \mathbb{R}^3 \mid A\vec{v} \leq \vec{b}\}$. Show that P is a polytope and find all its vertices.

   What is the maximum value of $x + 2y + 3z$ on P?

   Solution. P is clearly a polyhedron because it is the intersection of the halfspaces defined by the inequalities. To see that it is bounded (and therefore, a polytope) observe that by the first three rows of A we get for every point $(x, y, z) \in P$ that $x, y, z \leq 1$. Moreover, $-x - y - z \leq 1$ and consequently $x \geq -x - y - 1 \geq -3$. Similarly, $y, z \geq -3$. This shows that P is bounded.

   To find the vertices of P we notice that three linearly independent rows of A are every three that do not contain the last two rows together. This leaves 7 options. These are the first three rows, the forth row and any two of the first three (there is a symmetry here) and finally, the last row and any two of the first three (here too there is symmetry).

   The first three rows of A suggest the vertex $(1, 1, 1)$. However, this point is not feasible, as it does not satisfy the inequality of the forth line of A.

   Considering the forth row of A and the first two, we get the point $(1, 1, -1)$. This is a feasible point and therefore a vertex. Similarly, $(1, -1, 1)$ and $(-1, 1, 1)$ are vertices.

Considering the fifth row of A and the first two, we get the point $(1, 1, -3)$. This is a feasible point and therefore a vertex. Similarly, $(1, -3, 1)$ and $(-3, 1, 1)$ are vertices.

These are all the six vertices of P.

The objective function $x + 2y + 3z$ is maximized on a vertex of P. The vertex yielding the maximal objective value among all vertices is $(-1, 1, 1)$.

2. Find a hyperplane separating the ellipsoid $E = \{(x, y, z) \mid 2x^2 + \frac{2y^2}{4} + \frac{2z^2}{9} \leq 2\}$ from the point $p = (1, 2, 3)$.

   Solution. We present a systematic solution (not necessarily the shortest). The linear transformation $T(x, y, z) = (x, y/2, z/3)$ takes E to the unit ball in $\mathbb{R}^3$. T takes the point $(1, 2, 3)$ to the point $(1, 1, 1)$. It is not difficult to find a hyperplane separating the unit ball from $(1, 1, 1)$. For example $H = \{x + y + z = 2\}$ will do (why? - one can change 2 with any number strictly between $\sqrt{3}$ and 3).

   Now we only need to apply the inverse map T^{-1} on H to get a separating hyper-plane. T^{-1} is defined by $T^{-1}(x, y, z) = (x, 2y, 3z)$. T^{-1} takes H to the hyperplane $\{x + \frac{y}{2} + \frac{z}{3} = 2\}$.

3. Let A be the $2^n \times n$ matrix whose rows are all the 2^n possible combinations of 1 and -1. Let $\vec{b} = (1, 1, 1, \ldots, 1) \in \mathbb{R}^{2^n}$.

   Show that $\{\vec{x} \mid A\vec{x} \leq \vec{b}\}$ is a polytope and find all its vertices.

   Solution. Observe that if $(x_1, \ldots, x_n) \in P$, then necessarily $|x_1| + \ldots + |x_n| \leq 1$. This is because we can choose the inequality defined by the row of A in which the $+1$'s and -1's correspond to the signs of the coordinates $x_1, \ldots, x_n$. In particular, P is bouneded and therefore it is a polytope.

   We notice also that if $|x_1| + \ldots + |x_n| \leq 1$. then $(x_1, \ldots, x_n)$ is in P. Therefore, P is precisely the set of all points $(x_1, \ldots, x_n)$ with $|x_1| + \ldots + |x_n| \leq 1$. Therefore, any point on the boundary of P and in particular the vertices of P must satisfy $|x_1| + \ldots + |x_n| = 1$.

   If $(x_1, \ldots, x_n)$ is a point with $|x_1| + \ldots + |x_n| = 1$ and **both** $|x_i|$ **and** $|x_j|$ **are greater than** 0, then we may increase the absolute value of x_i and at the same time decrease by the same amount the absolute value of x_j and remain in P. We can also do the opposite operation and remain in P. Hence, there is a vector v such that both $\vec{x} + \vec{v}$ and

$\vec{x} - \vec{v}$ are in P. This implies that $\vec{x}$ cannot be a vertex. There cannot be a supporting hyperplane for P that meet P only at $\vec{x}$.

We conclude that the only possible candidates for vertices are the $2n$ points $(x_1, \ldots, x_n)$ with all the coordinates being equal to 0 except one coordinate that is equal either to 1 or -1. Because of symmetry, they are all vertices.

4. Suppose P is a polytope in $\mathbb{R}^n$. Let $T : \mathbb{R}^n \to \mathbb{R}^k$ be any linear map. Show that $T(P)$ is a polytope in $\mathbb{R}^k$.

   Solution. What can be the image of a half-space $B = \{\langle \vec{a}, \vec{x} \rangle \leq b\}$ under a linear transformation T? Let U denote the subspace $\{\langle \vec{a}, \vec{x} \rangle = 0\}$. Then $B = \cup_{t \leq b/\|\vec{a}\|^2}(t\vec{a} + U)$. Therefore, $T(B) = \cup_{t \leq b/\|\vec{a}\|^2}(tT(\vec{a}) + T(U)$.

   If $T(\vec{a}) \in T(U)$, then $T(U)$ is equal to the image of T. Otherwise $T(U)$ is a hyper-plane (of co-dimension 1) in the image of T and $T(B)$ is a half-space in the vector space that is the image if T. Denoting the image of T by W, we see that $T(P)$ is the intersection of half-spaces in W, or the entire space W. Because P is bounded and T is linear and consequently uniformly continuous, we conclude that $T(P)$ is bounded (because P is bounded). Therefore, $T(P)$ is a bounded intersection of half-spaces in W. We can present it also as an intersection of half-spaces in $\mathbb{R}^k$ by adding half-spaces of $\mathbb{R}^k$ whose intersection is equal to W. and extend each half-space of W to a half-space of $\mathbb{R}^k$.