Discrete Optimizaton 2023
Problem set of week 2 - Solutions

February 28, 2023

We say that a hyperplane $H = \{x \mid \langle x, a\rangle = b\}$ separates a point p from a set W, if $\langle p, a\rangle > b$ while $\langle x, a\rangle < b$ for every $x \in W$.

1. Find a point that is inside the tetrahedron whose facets are induced by the following supporting hyperplanes:
   $\{x+y+z = 1\}$, $\{2x-3y-z = 2\}$, $\{x-3y+z = 4\}$ and $\{2x-y+3z = 1\}$.

   Solution. One way is to find the four vertices of the tetrahedron. We do this by solving four times a system of three equations in three variables. This equivalent to intersecting each three of the hyperplanes. So, $A = h_1 \cap h_2 \cap h_3 = (0.5, -0.75, 1.25)^\top$, $B = h_1 \cap h_2 \cap h_4 = (8/7, 3/14, -5/14)^\top$, $C = h_1 \cap h_3 \cap h_4 = (5, -0.75, -13/4)^\top$ and $D = h_4 \cap h_2 \cap h_3 = (-1, -1.5, 0.5)^\top$.

   The point $\frac{1}{4}(A + B + C + D) = (\frac{79}{64}, -\frac{39}{64}, -\frac{13}{28})$ must be inside the tetrahedron since it is a convex combination of the vertices and the tetrahedron, as it is a polytope, is defined as the convex hull of its vertices. It is a convex combination since the sum of the four coefficients is $4 \cdot \frac{1}{4} = 1$ and also $0 \leqslant \frac{1}{4} \leqslant 1$. Furthermore, $\frac{1}{2}A + \frac{1}{2}(\frac{A+B+C}{3})$ is inside the tetrahedron for the same reasoning.

2. Find a hyperplane separating the point $(1, 2, 3)$ from the unit ball $B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leqslant 1\}$ in $\mathbb{R}^3$.

   Solution. We can take a hyperplane perpendicular to the line through the center of the ball, which is the origin $\mathbb{O}$, and the point $(1, 2, 3)$. Any such hyperplane passing through a point on the line interval between $(1, 2, 3)$ and $\frac{(1,2,3)}{|(1,2,3)|}$ will do.

   Hence, we can take $H = \{x + 2y + 3z = 10\}$, for example since this hyperplane satisfies that $(1, 2, 3)$ lies in the (open) halfspace $H^> = \{x +$

$2y + 3z > 10\}$ and any point of the ball B lies in the open halfspace $H^< = \{x + 2y + 3z < 10\}$, i.e., H is separating the halfspaces $(1, 2, 3)$ and B are lying in.

3. Let B be the cube $B = \{(x_1, \dots, x_6) \mid 0 \leqslant x_1, \dots, x_6 \leqslant 1\}$ in $\mathbb{R}^6$. Find a hyperplane passing through the point $(1, 0, 1, 0, 1, 0)$ that does not contain any other point of B.

   Solution. We can take a hyperplane perpendicular to the line through the center of the cube $\frac{1}{2}(1, 1, 1, 1, 1, 1)$ and the point $(1, 0, 1, 0, 1, 0)$.

   The reason why it will work is that the distance from $(1, 0, 1, 0, 1, 0)$ to the center of the cube is larger than or equal to the distance of any other point in the cube to the center of the cube.

   If the hyperplane we take contains another point of B, its distance to the center of the cube will be larger.

   Therefore, we take the hyperplane $H = \{x_1 - x_2 + x_3 - x_4 + x_5 - x_6 = 3\}$.

4. Let $p_1, \dots, p_m$ be m points in $\mathbb{R}^n$. Prove that the point $q = \frac{1}{m}(p_1 + \ldots + p_m)$ cannot be separated by a hyperplane from the points $p_1, \dots, p_m$.

   Solution. Assume the contrary, then there is a hyperplane $H = \{x \in \mathbb{R}^n : \langle x, a \rangle = b\}$ such that $\langle q, a \rangle < b$ while $\langle p_i, a \rangle > b$ for every $i = 1, \dots, m$. This is a contradiction because we get

$$b > \langle q, a \rangle = \sum_{i=1}^{m} \frac{1}{m} \langle p_i, a \rangle > \sum_{i=1}^{m} \frac{1}{m} b = b.$$