Integer Optimization
Problem Set 2

To be discussed on March 6

1. Show that a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$, with $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ is a convex set.

2. Prove Carathéodory's Theorem: Let $X \subseteq \mathbb{R}^n$, then for each $x \in \text{cone}(X)$ there exists a set $\widetilde{X} \subseteq X$ of cardinality at most n such that $x \in \text{cone}(\widetilde{X})$. The vectors in $\widetilde{X}$ are linearly independent.

3. Let $A \in \mathbb{R}^{n\times n}$ be a non-singular matrix and let $a_1,\dots,a_n \in \mathbb{R}^n$ be the columns of A. Show that $\text{cone}(\{a_1,\dots,a_n\})$ is the polyhedron $P = \{y \in \mathbb{R}^n\colon A^{-1}y \geqslant 0\}$. Show that $\text{cone}(\{a_1,\dots,a_k\})$ for $k \leqslant n$ is the set $P_k = \{y \in \mathbb{R}^n\colon a_i^{-1}x \geqslant 0, i = 1,\dots,k,\ a_i^{-1}x = 0, i = k+1,\dots,n\}$, where a_i^{-1} denotes the i-th row of A^{-1}.

4. Prove that for a finite set $X \subseteq \mathbb{R}^n$ the conic hull $\text{cone}(X)$ is closed and convex. Find a countably infinite set $X \subset \mathbb{R}^2$ such that $\text{cone}(X)$ is not closed.

5. A *vertex* of a polyhedron $P(A,b) \subseteq \mathbb{R}^n$ is an element $v \in P$ such that there do not exist $v_1, v_2 \in P$, $v_1 \neq v_2$ with $v = 1/2(v_1 + v_2)$. Show that a polyhedron $P(A,b)$ has a vertex if and only if $\text{rank}(A) = n$.

6. Let $P = P(A,b)$ be a rational polyhedron (meaning that both A and b can be chosen to be rational). Show that for each $c \in \mathbb{R}^n$ one has
$$\max\{c^T x : x \in P \cap \mathbb{Z}^n\} = \max\{c^T x : x \in P_I\}.$$

7. Show that each vertex of the polytope that is described by the inequalities
$$0 \leqslant x_i \leqslant 1, \quad i = 1,\dots,n$$
$$\sum_{i=1}^n x_i \leqslant \beta$$
with $\beta \in \mathbb{Z}$ is an integral point. Argue that the integer program
$$\max \sum_{i=1}^n c_i x_i$$
subject to
$$0 \leqslant x_i \leqslant 1, \quad i = 1,\dots,n$$
$$\sum_{i=1}^n 2\cdot x_i \leqslant n$$
$$x \in \mathbb{Z}^n$$
can be solved in polynomial time.