Integer Optimization
Problem Set 2

To be discussed on March 6

1. Show that a polyhedron \( P = \{ x \in \mathbb{R}^n : Ax \leq b \} \), with \( A \in \mathbb{R}^{m \times n} \), \( b \in \mathbb{R}^m \) is a convex set.

2. Prove Carathéodory’s Theorem: Let \( X \subseteq \mathbb{R}^n \), then for each \( x \in \text{cone}(X) \) there exists a set \( \tilde{X} \subseteq X \) of cardinality at most \( n \) such that \( x \in \text{cone}(\tilde{X}) \). The vectors in \( \tilde{X} \) are linearly independent.

3. Let \( A \in \mathbb{R}^{n \times n} \) be a non-singular matrix and let \( a_1, \ldots, a_n \in \mathbb{R}^n \) be the columns of \( A \). Show that \( \text{cone}(\{a_1, \ldots, a_n\}) \) is the polyhedron \( P = \{ y \in \mathbb{R}^n : A^{-1}y \geq 0 \} \).

4. Prove that for a finite set \( X \subseteq \mathbb{R}^n \) the conic hull \( \text{cone}(X) \) is closed and convex. Find a countably infinite set \( X \subseteq \mathbb{R}^2 \) such that \( \text{cone}(X) \) is not closed.

5. A vertex of a polyhedron \( P(A, b) \subseteq \mathbb{R}^n \) is an element \( v \in P \) such that there do not exist \( v_1, v_2 \in P \), \( v_1 \neq v_2 \) with \( v = 1/2(v_1 + v_2) \). Show that a polyhedron \( P(A, b) \) has a vertex if and only if \( \text{rank}(A) = n \).

6. Let \( P = P(A, b) \) be a rational polyhedron (meaning that both \( A \) and \( b \) can be chosen to be rational). Show that for each \( c \in \mathbb{R}^n \) one has
\[
\max\{ c^T x : x \in P \cap \mathbb{Z}^n \} = \max\{ c^T x : x \in P \}.
\]

7. Show that each vertex of the polytope that is described by the inequalities
\[
0 \leq x_i \leq 1, \quad i = 1, \ldots, n
\]
\[
\sum_{i=1}^n x_i \leq \beta
\]
with \( \beta \in \mathbb{Z} \) is an integral point. Argue that the integer program
\[
\max \sum_{i=1}^n c_i x_i
\]
subject to
\[
0 \leq x_i \leq 1, \quad i = 1, \ldots, n
\]
\[
\sum_{i=1}^n 2 \cdot x_i \leq n
\]
x \in \mathbb{Z}^n

can be solved in polynomial time.