Integer Optimization
Problem Set 1

To be discussed on Feb. 27.

1. An employment agency wants to find a job for a set of workers, W. The set of available jobs is E. The agency knows whether a worker $i \in W$ is qualified for job $j \in E$. Describe an integer program to find the maximum number of workers being employed in jobs for which they are qualified. *Hint: Use variables $x_{i,j} \in \{0,1\}$ indicating whether worker i was assigned job j.*

   Now, some companies offering jobs, have a maximum number of new hires. For example, the subset $S \subseteq E$ of jobs is offered by IBN. IBN can, for budget reasons, only hire 5 new people. Incorporate these additional constraints into your integer program.

2. Recall that for $X \subseteq \mathbb{R}^n$ the convex hull of X is

$$\operatorname{conv}(X) = \left\{ \sum_{i=1}^{t} \mu_i x_i : \ t \in \mathbb{N}_+, \mu_i \geqslant 0, x_i \in X, \sum_{i=1}^{t} \mu_i = 1 \right\}.$$

   Show that, for $A, B \subseteq \mathbb{R}^n$, one has $\operatorname{conv}(A \cup B) = \operatorname{conv}(\operatorname{conv}(A) \cup \operatorname{conv}(B))$.

3. Recall that for $X \subseteq \mathbb{R}^n$, $X_I = \operatorname{conv}(X \cap \mathbb{Z}^n)$.

   Let $\mathscr{C} = \operatorname{cone}(\{c_1, \ldots, c_r\})$ with $c_i \in \mathbb{Z}^n$ for $1 \leqslant i \leqslant r$. Show that $\mathscr{C}_I = \mathscr{C}$.

   Hint: Write each element of $\mathscr{C}$ as a convex combination of integer multiples of the c_i and 0.

4. In this exercise, you can assume that each cone, $\operatorname{cone}(X)$ with $X \subseteq \mathbb{R}^n$ finite can be written as $P(A,0) = \{x \in \mathbb{R}^n : \ Ax \leqslant 0\}$ for some matrix A and vice versa.

   Prove the Minkowski-Weyl theorem: A set $P \subseteq \mathbb{R}^n$ is a polyhedron if and only if there exist finite sets $X, Y \subseteq \mathbb{R}^n$ with $P = X + Y$.

   Hint: Let $P = P(A,b)$. Consider the polyhedral cone $\mathscr{C} = \{\binom{x}{\lambda} \in \mathbb{R}^{n+1} : \ Ax \leqslant \lambda b, \lambda \geqslant 0\}$.

5. Let $P = \{x \in \mathbb{R}^2 : (1,\alpha)^T x \leqslant 0\}$. If $\alpha \notin \mathbb{Q}$, then P_I is not a polyhedron.

6. Consider the triangle $T \subseteq \mathbb{R}^2$ with vertices $(0,0), (a,0), (a,\gamma)$ where $a \in \mathbb{N}$ and $\gamma \in \mathbb{R}_{\geqslant 0}$. Show that there exists a set $S \subseteq (T \cap \mathbb{Z}^2)$ with $|S| = O(\log a)$ such that $T_I = \operatorname{conv}(S)$ holds.