Integer Optimization
Problem Set 1

To be discussed on Feb. 27.

1. An employment agency wants to find a job for a set of workers, $W$. The set of available jobs is $E$. The agency knows whether a worker $i \in W$ is qualified for job $j \in E$. Describe an integer program to find the maximum number of workers being employed in jobs for which they are qualified. *Hint: Use variables $x_{i,j} \in \{0,1\}$ indicating whether worker $i$ was assigned job $j$.*

Now, some companies offering jobs, have a maximum number of new hires. For example, the subset $S \subseteq E$ of jobs is offered by IBN. IBN can, for budget reasons, only hire 5 new people. Incorporate these additional constraints into your integer program.

2. Recall that for $X \subseteq \mathbb{R}^n$ the convex hull of $X$ is

$$
\text{conv}(X) = \left\{ \sum_{i=1}^{t} \mu_i x_i : \mu_i \geq 0, x_i \in X, \sum_{i=1}^{t} \mu_i = 1 \right\}.
$$

Show that, for $A, B \subseteq \mathbb{R}^n$, one has $\text{conv}(A \cup B) = \text{conv}(\text{conv}(A) \cup \text{conv}(B))$.

3. Recall that for $X \subseteq \mathbb{R}^n$, $X_I = \text{conv}(X \cap \mathbb{Z}^n)$.

Let $\mathcal{C} = \text{cone}(\{c_1, \ldots, c_r\})$ with $c_i \in \mathbb{Z}^n$ for $1 \leq i \leq r$. Show that $\mathcal{C}_I = \mathcal{C}$.

*Hint: Write each element of $\mathcal{C}_I$ as a convex combination of integer multiples of the $c_i$ and 0.*

4. In this exercise, you can assume that each cone, $\text{cone}(X)$ with $X \subseteq \mathbb{R}^n$ finite can be written as $P(A,0) = \{x \in \mathbb{R}^n : Ax \leq 0\}$ for some matrix $A$ and vice versa.

Prove the Minkowski-Weyl theorem: A set $P \subseteq \mathbb{R}^n$ is a polyhedron if and only if there exist finite sets $X, Y \subseteq \mathbb{R}^n$ with $P = X + Y$.

*Hint: Let $P = P(A,b)$. Consider the polyhedral cone $\mathcal{C}' = \{(x,\lambda) \in \mathbb{R}^{n+1} : Ax \leq \lambda b, \lambda \geq 0\}. $

5. Let $P = \{x \in \mathbb{R}^2 : (1, \alpha)^T x \leq 0\}$. If $\alpha \notin \mathbb{Q}$, then $P_I$ is not a polyhedron.

6. Consider the triangle $T \subseteq \mathbb{R}^2$ with vertices $(0,0), (a,0), (a, \gamma)$ where $a \in \mathbb{N}$ and $\gamma \in \mathbb{R}_{\geq 0}$. Show that there exists a set $S \subseteq (T \cap \mathbb{Z}^2)$ with $|S| = O(\log a)$ such that $T_I = \text{conv}(S)$ holds.