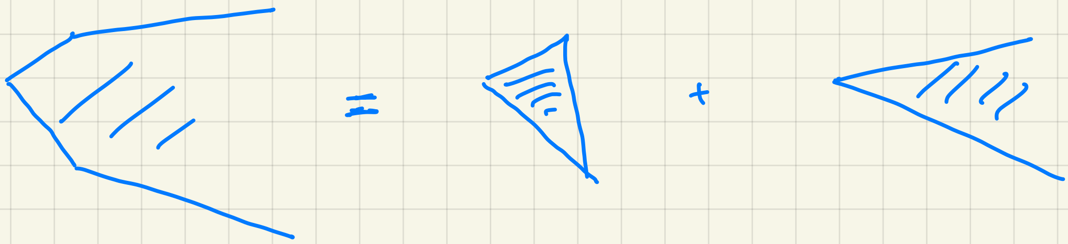
# Lecture 2, Monday 27. Feb. 2023

**Recap:** $P = P(A,b) \subseteq \mathbb{R}^n$ polyhedron, then

$$P = \operatorname{conv}\{v_1, \dots, v_\ell\} \oplus \operatorname{cone}\{r_1, \dots, r_k\}$$

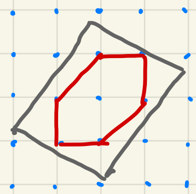


If $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$, then one can assume

$v_i \in \mathbb{Q}^n$ $\quad i = 1, \dots, \ell$ and

$r_j \in \mathbb{Z}^n$ $\quad j = 1, \dots, k$.

**Def:** For $X \subseteq \mathbb{R}^n$, $(X)_I = \operatorname{conv}(X \cap \mathbb{Z}^n)$ <u>integer hull</u> of $X$.

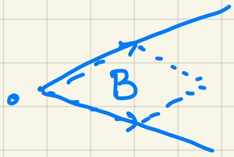


**Thm:** If $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$, then $[P(A,b)]_I$ is a polyhedron.

**Proof:** $P(A,b) = Q \oplus \mathcal{C}$ with $Q = \operatorname{conv}\{v_1, \dots, v_k\}$

$\mathcal{C} = \operatorname{cone}\{r_1, \dots, r_\ell\}$

each $r_i$ integral.




Let $B = \left\{ \sum_{j=1}^{\ell} \mu_j \cdot r_j \; : \; 0 \le \mu \le 1 \right\}$

**Claim:** $P_I = (Q \oplus B)_I \oplus \mathcal{C}$

**Proof of Claim:** "$\subseteq$" We show: ech $p \in P \cap \mathbb{Z}^n$ is in

$$(Q \oplus B)_I \oplus \mathcal{C}.$$

$$p = q + c \qquad q \in Q, \quad c \in \mathcal{C}$$

$$c = \sum_{j=1}^{\ell} \lambda_j \cdot r_j = \underbrace{\sum_{j=1}^{\ell} \{\lambda_j\} \cdot r_j}_{= b \in B} + \underbrace{\sum_{j=1}^{\ell} \lfloor \lambda_j \rfloor \cdot r_j}_{= c' \in \mathcal{C} \cap \mathbb{Z}^n}$$

with $\lambda \ge 0$

$$p = \underbrace{q + b}_{\in (Q \oplus B)_I} + \underbrace{c'}_{\in \mathcal{C}} \qquad (\checkmark)$$

„$\supseteq$"  $(Q+B)_I + \mathscr{C}$

$\subseteq P_I + \mathscr{C}$

$= P_I + \mathscr{C}_I$

Exercise  $\subseteq (P + \mathscr{C})_I$

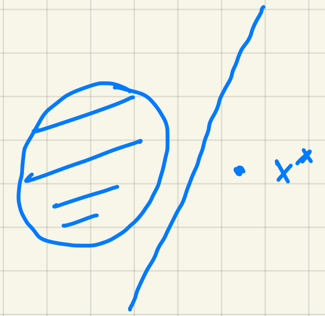
$= P_I$

$\blacksquare$

# Proof of Minkowski-Weyl Theorem

<u>Separation Thm:</u> Let $C \subseteq \mathbb{R}^n$ be closed and convex and $x^* \in \mathbb{R}^n \setminus C$. There exist $a \in \mathbb{R}^n \setminus \{0\}$ and $\beta \in \mathbb{R}$ s.th.

$$a^T \cdot x^* > \beta$$
$$a^T \cdot x < \beta \quad \forall x \in C$$



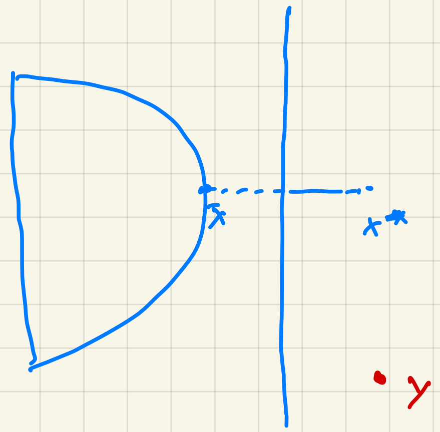


<u>Proof:</u> (sketch) Assume first that $C$ is compact.

$f: C \to \mathbb{R}$

$x \mapsto \|x - x^*\|$ continuous. Thus attains min

Let min be attained at $\bar{x} \in C$




consider hyperplane through $\frac{1}{2}(\bar{x} + x^*)$ with normal vector $(x^* - \bar{x})$

if $\exists\, y \in C$ on same side as $x^*$

then line-segment $\overline{y, \bar{x}} \subseteq C$

points close to $\bar{x}$ on line-segment are closer to $x^*$ than $\bar{x}$ ↯.

$C'$ unbounded.

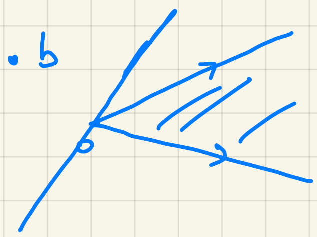
Let $\tilde{x} \in C$ arbitrary.

Consider $C \cap B$ $\quad B = \{x \in \mathbb{R}^n \mid \|x^* - x\| \leq \|x^* - \tilde{x}\|\}$

and repeat argument above to find $\bar{x}$.

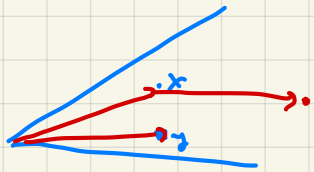
<u>Consequence:</u> <u>Farkas' Lemma</u> Let $X \subseteq \mathbb{R}^n$ finite and $b \in \mathbb{R}^n$. $b \notin \text{cone}(X) \iff \exists\, a \in \mathbb{R}^n \neq 0$ s.th.

$a^T \cdot b > 0 \quad a^T \cdot x^* \leq 0 \quad \forall x^* \in X$




<u>Proof:</u> cone$(X)$ is closed and convex.

**Def:** $\mathcal{C} \subseteq \mathbb{R}^n$ is called convex cone if $\forall\, x,y \in \mathcal{C}$ and $\forall\, \lambda,\mu \in \mathbb{R}_{\geq 0}$ : $\lambda \cdot x + \mu \cdot y \in \mathcal{C}$




$\mathcal{C}$ is polyhedral if $\mathcal{C} = \{x : A \cdot x \leq 0\}$ for some $A \in \mathbb{R}^{m \times n}$

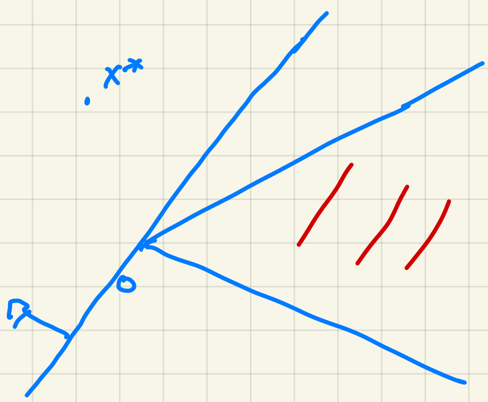
$\mathcal{C}$ is finitely generated if $\mathcal{C} = \text{cone}(X)$ for some $X \subseteq \mathbb{R}^n$ finite.

**Thm:** A convex cone is polyhedral if and only if it is finitely generated.

**Proof:** "$\Leftarrow$" $\mathcal{C} = \text{cone}\{a_1, \dots, a_m\} \subseteq \mathbb{R}^n$

Case 1: $\text{span}\{a_1, \dots, a_m\} = \mathbb{R}^n$

$c^T \cdot x = 0$




$c^T \cdot x^* > 0$

$c^T \cdot a_i \leq 0$

$I = \{i : c^T \cdot a_i = 0\}$.

$\dim\{a_i : i \in I\} < n-1$ $\quad u \in \mathbb{R}^n \setminus \{0\}$

$u \perp a_i \ i \in I, \quad u \perp x^*$

$J = \{ j : u^T \cdot a_j > 0 \}$ w.l.o.G. $J \neq 0$.

$\forall j \in J$: $(c + \varepsilon u^T)\, a_j \leq 0$ $\varepsilon \in \mathbb{R}_{\geq 0}$ maximal.

$$\varepsilon \leq \frac{-c^T \cdot a_j}{u^T \cdot a_j}$$

$$\varepsilon^* = \min \left\{ \frac{-c^T \cdot a_j}{u^T \cdot a_j} : j \in J \right\}$$

$\Rightarrow$ assume $\dim \{ a_i : i \in I \} = n-1$

<u>Case 2:</u> $\dim \langle a_1, \dots, a_m \rangle = n - k$

$\Rightarrow \exists$ $d_1, \dots, d_k$ orthogonal to $\langle a_1, \dots, a_m \rangle$

s.th. $\langle a_1, \dots, a_m, d_1, \dots, d_k \rangle = \mathbb{R}^n$

$\operatorname{cone}(\{a_1,\dots,a_m, d_1,\dots,d_r\}) = \{x : Ax \le 0\}.$

$\langle a_1,\dots,a_m \rangle = \{x \in \mathbb{R}^n : Ux = 0\}$

$\Rightarrow \quad \operatorname{cone}\{a_1,\dots,a_m\} = \{x : Ax \le 0,\ Ux = 0\}.$

"$\Rightarrow$"

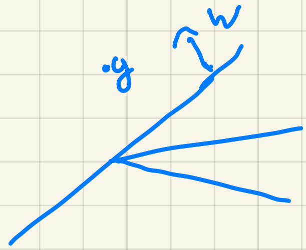
$C = \{x : a_1^T x \le 0, \dots, a_m^T x \le 0\}.$

$C' = \{\sum \lambda_i a_i : \lambda_i \ge 0\}.$

$C' = \{x \in \mathbb{R}^n : b_1^T x \le 0, \dots, b_r^T x \le 0\}$

$\operatorname{cone}\{b_1,\dots,b_r\} \subseteq C$ since $a_i^T b_j \le 0$

Suppose $\exists\, y \in C \setminus \operatorname{cone}\{b_1,\dots,b_r\}.$




$\forall i:\ w^T b_i \le 0$ and $w^T y > 0$

$\Downarrow$

$w \in C'$ $\qquad$ $y \in C$ and $w = \sum \lambda_i a_i$

$\Rightarrow y^T w \le 0 \quad ↯.$

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