Integer Optimization

1. What is an integer program?

$$\max \; c^T \cdot x$$
$$Ax \le b$$
$$x \in \mathbb{Z}^n$$

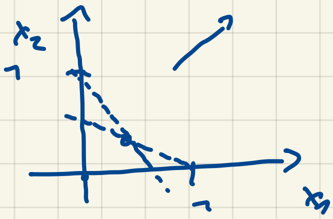
$x \in \mathbb{R}^n$
Linear program
efficient

Example: $\max \; x_1 + x_2$

$$2 \cdot x_1 + x_2 \le 1$$
$$x_1 + 2x_2 \le 1$$
$$x \ge 0$$

$OPT = 0$

$OPT_{LP} = 2/3$



Example: SAT $\quad \phi = C_1 \wedge \dots \wedge C_m$

literals

$$C_i = \{ l_1^i, \dots, l_{k_i}^i \}$$

literal $l = x_j$ or $l = \overline{x_j}$ $\quad j \in \{1, \dots, n\}$

n is number of variables

$$\phi = \{x_1, \overline{x_3}, x_5\} \wedge \{\overline{x_2}, x_5\} \wedge \{x_3, \overline{x_5}\}$$

Truth assignment: $f : \{1, \dots, n\} \to \{0, 1\}$

$f \models \phi$ if $\forall i \in \{1, \dots, m\}$ $\exists j \in \{1, \dots, k_i\}$ s.th. $f \models l_j^i$

$f \models l$ if $l = x_j$ and $f(j) = 1$ or $l = \overline{x_j}$ and $f(j) = 0$

Represent f as $x \in \{0,1\}^n$

$$A_{ij} = \begin{cases} 1 & \text{if } x_j \text{ lit in } C_i \\ -1 & \text{if } \overline{x_j} \text{ — " —} \\ 0 & \text{oth.} \end{cases}$$

$b_i = 1 + n_i$ $\quad\quad$ n_i is number of neg literals in C_i

ϕ is sat. $\Leftrightarrow$ $\begin{array}{c} Ax \geq b \\ x \in \{0,1\}^n \end{array}$ feasible.

<u>Ex:</u> $\phi = \{x_1, \bar{x}_3, \bar{x}_5\} \wedge \{\bar{x}_2, x_6\} \wedge \{x_3, \bar{x}_5\}$

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}$$

$$b = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$$

<u>Example:</u> Set Cover

$\mathcal{U} = \{1, \dots, m\}$ $\quad$ $S_1, \dots, S_n \subseteq \mathcal{U}$ with $\bigcup S_i = \mathcal{U}$

$I \subseteq \{1, \dots, n\}$ is set-cover if $\bigcup_{i \in I} S_i = \mathcal{U}$

<u>Task:</u> $\min\limits_{I \subseteq \{1,\dots,n\} \text{ set cover}} |I|$

$$A = \begin{bmatrix} | & & | \\ \chi_{S_1} & \cdots & \chi_{S_n} \\ | & & | \end{bmatrix} \in \{0,1\}^{m \times n}$$

$A_{ij} = \begin{cases} 1 & \text{if } i \in S_j \\ 0 & \text{oth.} \end{cases}$ $\quad\quad$ $\min \sum x_i$

$A \cdot x \geq \mathbb{1}$

$\left.\begin{array}{l} x \geq 0 \\ x \in \mathbb{Z}^n \end{array}\right\} \Leftrightarrow x \in \{0,1\}^n.$

Example: gcd(a,b)

$a, b \in \mathbb{Z}$ not both zero

$$\gcd(a,b) = \min \{x \cdot a + y \cdot b : x \cdot a + y \cdot b \geq 1, \; x, y \in \mathbb{Z}\}$$

IP: $\min \; (a,b)\begin{pmatrix} x \\ y \end{pmatrix}$

$$(a,b)\begin{pmatrix} x \\ y \end{pmatrix} \geq 1$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^2$$

IP with two variables.

Unimodular transformations:

$(\mathbb{Z}^n, +)$ is (additive) abelian group.

Let $A \in \mathbb{Z}^{n \times n}$ be a matrix and $\varphi : \mathbb{Z}^n \to \mathbb{Z}^n$

$x \mapsto A \cdot x$

Proposition: φ is automorphism $\iff \det(A) = \pm 1$

Proof: „$\Rightarrow$" $\varphi(u_i) = e_i \qquad u_i \in \mathbb{Z}^n$

$$A\,(u_1, \dots, u_n) = I$$

$\Rightarrow \det(A) = \pm 1$

„$\Leftarrow$" φ is injective $(\ker(A) = \{0\})$

surjective $\quad A^{-1} = \frac{\tilde{A}}{\det(A)} \in \mathbb{Z}^{n \times n}$.

Consequence: If $U \in \mathbb{Z}^{n \times n}$ unimodular, then

$$\begin{array}{ll} \max & c^T \cdot x \\ & Ax \le b \\ & x \in \mathbb{Z}^n \end{array} \quad \equiv \quad \begin{array}{ll} \max & c^T \cdot U \cdot x \\ & A \cdot U \cdot x \le b \\ & x \in \mathbb{Z}^n \end{array}$$

$\uparrow$

equivalent

Example:

$$\max \quad x_1 + x_2$$

$$3 \cdot x_1 + x_2 \le 1$$

$$x_1 + 3 \cdot x_2 \le 1$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{Z}^2$$

$$(\equiv) \quad \max \quad \mathbb{1}^T \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \le \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$(\equiv) \quad \max \quad (1, -2)^T \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

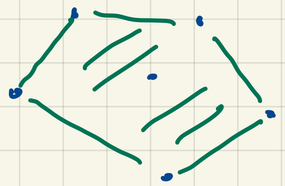
$$\begin{pmatrix} 3 & -8 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \le \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -8 \\ 1 & 0 \end{pmatrix}$$

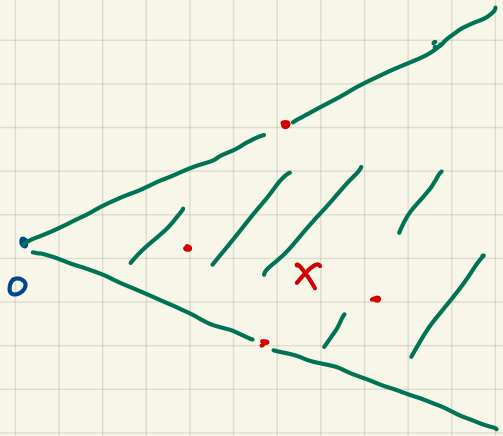
Convex and Conic Hulls:

For $X \subseteq \mathbb{R}^n$

$$\mathrm{conv}(X) = \left\{ \sum_{i=1}^{t} \lambda_i \cdot x_i \;:\; t \in \mathbb{N}_+ ,\ x_i \in X \ i=1,\dots,t \right.$$
$$\left. \lambda \geq 0,\ \sum_{i=1}^{t} \lambda_i = 1 \right\}$$



$$\mathrm{cone}(X) = \left\{ \sum_{i=1}^{t} \lambda_i \cdot x_i \;:\; t \in \mathbb{N}_+ ,\ x_i \in X \ i=1,\dots,t, \right.$$
$$\left. \lambda_i \geq 0 \right\}$$



Polyhedra: Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

$$P(A,b) = \{ x \in \mathbb{R}^n : Ax \leq b \}$$

is called Polyhedron.

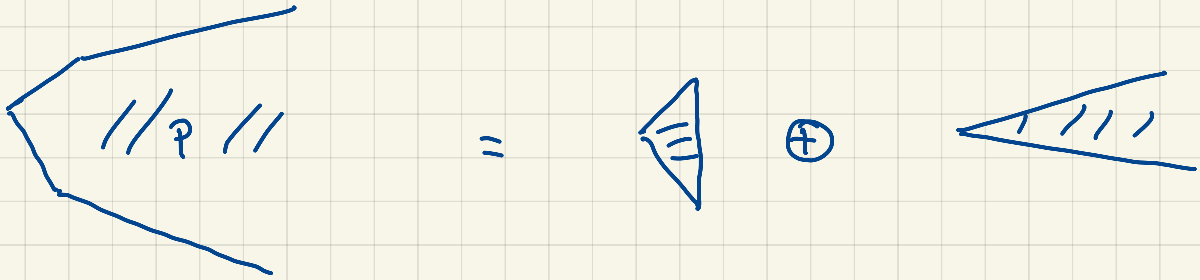
Minkowski-Sum: $A, B \subseteq \mathbb{R}^n$

$$A \oplus B = \{ a + b : a \in A,\ b \in B \}.$$

Theorem: (Minkowski-Weyl) $P \subseteq \mathbb{R}^n$ is polyhedron

iff $\exists$ finite sets $V, R \subseteq \mathbb{R}^n$ s.th.

$$P = \operatorname{conv}(V) + \operatorname{cone}(R)$$



Furthermore: For $P(A, b)$

n rows of A

each $v \in V$ is solution of $A_B x = b_B$

with $\operatorname{rank}(A_B) = n$

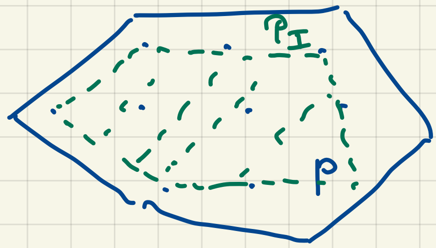
$r \in R$ —"— $A_I x = 0$

with $\operatorname{rank}(A_I) = n-1$

The integer hull:

Let $X \subseteq \mathbb{R}^n$

$$X_I = \operatorname{conv}(X \cap \mathbb{Z}^n)$$



Thm: If $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, then

$P(A,b)_I$ is polyhedron.

proof: $P = \underset{\overset{\|}{Q}}{\mathrm{conv}(V)} + \underset{\overset{\|}{\mathcal{C}}}{\mathrm{cone}(R)}$

$R \subseteq \mathbb{Z}^n \qquad R = \{r_1, \dots, r_k\}$

Let $B = \left\{ \sum_{i=1}^{k} \lambda_i r_i \;:\; 0 \le \lambda_i \le 1 \right\}$

Claim $P_I = (Q+B)_I + \mathcal{C}$

"$\subseteq$" We show that each $v \in P \cap \mathbb{Z}^n$ is in $(Q+B)_I + \mathcal{C}$

$v = q + c$ and $c = b + c'$, $b \in B$, $c' \in \mathcal{C}_{\cap \mathbb{Z}^n}$

$= \underbrace{q + b}_{\in \mathbb{Z}^n} + c' \in (Q+B)_I + \mathcal{C}$

"$\supseteq$" $(Q+B)_I + \mathcal{C} \subseteq P_I + \mathcal{C} = P_I + \mathcal{C}_I$

$\subseteq (P + \mathcal{C})_I$

$= P_I$

$\blacksquare$

Consequently: If A, b rational, then.

$$\begin{matrix} \max c^T \cdot x \\ Ax \le b \\ x \in \mathbb{Z}^n \end{matrix} \quad = \quad \begin{matrix} \max c^T \cdot x \\ A' \cdot x \le b' \\ x \in \mathbb{R}^n \end{matrix}$$

with $P(A', b') = P_I$

LINEAR PROGRAM.