Discrete Optimization 2023
Problem set week 1 – Solutions

February 21, 2023

1. Show that the three medians in a triangle with vertices v_1, v_2, v_3 meet at the point $\frac{1}{3}(v_1 + v_2 + v_3)$.
Solution: The mid point of the edge $v_1 v_2$ is $\frac{v_1+v_2}{2}$. Notice that $\frac{1}{3}(v_1 + v_2 + v_3) = \frac{1}{3}v_3 + \frac{2}{3}(\frac{v_1+v_2}{2})$. Therefore, the point $\frac{1}{3}(v_1 + v_2 + v_3)$ lies on the median going from v_3. By symmetry it also lies on the other two medians.

2. Find the hyperplane passing through $(1, 1, 1)$ that is perpendicular to both hyperplanes $\{x + 2y + z = 2\}$ and $\{x - y - 3z = 8\}$ in $\mathbb{R}^3$.
Solution: In $\mathbb{R}^3$ a hyperplane equation is given by $ax+by+cz = d$ where $a, b, c, d \in \mathbb{R}$. We are looking for a triple (a, b, c) that is perpendicular to both $(1, 2, 1)$ and $(1, -1, -3)$. We calculate that a vector perpendicular to the both of them is $(5, -4, 3)$ or multiples of it since it satisfies both $(5, -4, 3)^T \cdot (1, 2, 1) = 0$ and $(5, -4, 3)^T \cdot (1, -1, -3) = 0$. Furthermore, $(1, 1, 1)$ needs to be on the hyperplane by assumptions. Thus, we obtain $d = 5 \cdot 1 - 4 \cdot 1 + 3 \cdot 1 = 4$. Hence, the hyperplane we searched for is $\{5x - 4y + 3z = 4\}$.

3. Find the closest point to $(3, 5, 4)$ on the hyperplane $\{2x + 4y - z = 3\}$ in $\mathbb{R}^3$
Solution: The geometric intuition of finding such point is, going along the normal vector of the hyperplane with start from our given point, until we reach the hyperplane.
There are 2 ways to solve this problem:

    (1) Let the point be $(a, b, c) \in \mathbb{R}^3$. We want the difference to be perpendicular to the hyperplane, which is equivalent to say that the difference is parallel to the normal vector of the hyperplane, that is,

    $$(a, b, c) - (3, 5, 4) \quad \text{is a multiple of} \quad (2, 4, -1)$$

We also need the point to be on the hyperplane, i.e., $2a + 4b - c = 3$. Solve those equations to get $a = \frac{25}{21}, b = \frac{29}{21}, c = \frac{103}{21}$.

(2) The point we are looking for can be represented as $(3,5,4)+t(2,4,-1)$. We also require that it's on the hyperplane, that is,

$$2(3+2t) + 4(5+4t) - (4-t) = 3 \Longrightarrow t = -\frac{19}{21}.$$

Bring it back to the expression will give us the same solution as the 1st method.

4. Find the distance of the origin $\mathbb{O}$ to the line of the intersection of the hyperplanes $\{x + y + z = 1\}$ and $\{2x - y + 3z = 1\}$ in $\mathbb{R}^3$.
Solution:

(1) First we compute the direction of the direction of the intersection line of these 2 hyperplanes. Since it is in the direction perpendicular to both $(1,1,1)$ and $(2,-1,3)$, we may take $(4,-1,-3)$ or any constant multiple of it.

(2) Then we find a point on the line of intersection of these 2 hyperplanes (take any point as you want), say $(1, \frac{1}{4}, -\frac{1}{4})$.

(3) The point on the plane with the shortest distance to the origin is of the form $(1, \frac{1}{4}, -\frac{1}{4}) + t(4,-1,-3)$, for some $t \in \mathbb{R}$. And this point (as a vector) should be perpendicular to the direction of the line, i.e., to $(4,-1,-3)$. Solve this to get $t = -\frac{9}{52}$.

(4) The distance will then be the distance from the origin to the point $(\frac{4}{13}, \frac{11}{26}, \frac{7}{26})$, which is $\frac{3\sqrt{26}}{26}$.