Metric Embeddings
Problem Set 3
Exercises on Measure Concentration

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In the following you can assume that sampling a point on the *unit sphere* $S^{n-1} = \{x \in \mathbb{R}^n \colon \|x\|_2 = 1\}$ at random can be done by sampling $x \in \mathbb{R}^n$ from the *standard n-dimensional Gaussian* with density function $(1/2\pi)^{n/2} \exp(-\|x\|_2^2/2)$ and returning $x/\|x\|_2$. This corresponds to sampling the n components of the vector x from the standard Gaussian with mean 0 and variance 1. You will need the theorem that was stated in today's lecture which said that the standard n-dimensional Gaussian is sharply concentrated around the sphere of radius $\sqrt{n}$, i.e., the probability that $\|x\|$ is not in the interval $[(1-\varepsilon)\sqrt{n}, (1+\varepsilon)\sqrt{n}]$ is bounded by $e^{c \cdot n \cdot \varepsilon^2}$.

1. Let $x, y \in S^{n-1}$ be two random points. What is the expected angle between the two?

2. This series of questions shall lead us to a proof that almost all measure of S^{n-1}, 90% say, is concentrated around the *equator* $E = \{x \in \mathbb{R}^n \colon x_1 = 0\}$. More precisely we show that 90% of the measure is within the slab $L = \{x \in \mathbb{R}^n \colon -C/\sqrt{n} \leqslant x_1 \leqslant C/\sqrt{n}\}$ where C is some constant.

   i) What is the diameter of S^{n-1}? It seems that $L \cap S^{n-1}$ "looks" like a small portion of S^{n-1}.

   ii) Assume for now (wishful thinking that is of course wrong!) that a random $x \in \mathbb{R}^n$ from the n-dimensional standard Gaussian has length $\|x\| = \sqrt{n}$. The distance $d(x)$ of $x/\|x\|$ to the equator is thus $x_1/\sqrt{n}$. Show that there exists a constant a such that

   $$P\left[d(x) > \lambda/\sqrt{n}\right] \leqslant \exp(-a \cdot \lambda^2).$$

3. Use our measure concentration for the Gaussian measure to show that there exists a constant C such that 90% of the measure of S^{n-1} lies in the slab L.

4. Derive a good upper bound for the event that a random $x \in S^{n-1}$ satisfies $d(x) \geqslant \lambda/\sqrt{n}$.