Metric Embeddings
Problem Set 2

October 7, 2022

These problems are discussed during the exercise session on Friday, October 7.

1) a) Let $K \subseteq \mathbb{R}^n$ be a centrally symmetric (around 0) convex, compact and full-dimensional set. Show that $\|v\|_K = \max\{t\colon\ t \cdot v \in K\}^{-1}$ is a norm. This norm is called ℓ_K.

   b) Show that, if K is in addition a polytope described as $K = \{x \in \mathbb{R}^n\colon -\mathbb{1} \leqslant Ax \leqslant \mathbb{1}\}$, with $A \in \mathbb{R}^{m\times n}\}$, then ℓ_K embeds isometrically into ℓ_∞^m.

2) A finite metric (X, d) is called a metric of *negative type*, if $(X, \sqrt{d})$ is ℓ_2 embeddable.

   (a) Show that a cut metric is a metric of negative type.

   (b) If (X, d_1) and (X, d_2) are of negative type and if $\alpha, \beta \in \mathbb{R}_{>0}$, then $(X, \alpha d_1 + \beta d_2)$ are of negative type.

   (c) Conclude that ℓ_1 is of negative type.

3) a) Let $X \subseteq \mathbb{R}^n$ be a finite set. Show that $d(u, v) = \|u - v\|_2^2$ yields a metric on X if and only if the angle between any three points $u, v, w \in X$ satisfies $\angle u, v, w \leqslant \pi/2$, i.e., no angle is *obtuse*.

   b) Show that this is the case for $X = \{0,1\}^n$, i.e. the vertices of the hypercube.

   There is a matching upper bound: If a finite set $X \subseteq \mathbb{R}^n$ does not permit obtuse angles, the $|X| \leqslant 2^n$. It is not part of the exercise.

   c) Conclude that the dimension of the ℓ_2-embedding of $(X, \sqrt{d})$, where (X, d) is of negative type, is at least $\log_2 |X|$.

4) A finite semi-metric (X, d), $X = \{1, \ldots, n\}$ is isometrically embeddable into ℓ_2^k if and only if the matrix $a(i, j), 1 \leqslant i, j \leqslant n-1$ as defined below is positive semidefinite and of rank at most k.

$$a_{ij} = 1/2\left(d^2(i,n) + d^2(j,n) - d^2(i,j)\right).$$