Metric Embeddings
Problem Set 1

September 20, 2022

These problems are discussed during the exercise session on Friday, September 30.

1) a) Show that every embedding of an n-point equilateral space (every two points have distance 1) into the plane $\mathbb{R}^2$ with the usual Euclidean metric has distortion at least $\Omega(\sqrt{n})$.

   b) Provide an embedding with distortion $O(\sqrt{n})$.

2) Show that every embedding of the cycle of length n (with the graph metric) into the line $\mathbb{R}$ with the usual metric has distortion at least $\Omega(n)$.

3) Show that every finite tree metric space can be embedded isometrically into ℓ_1. (You can start with embedding all trees with unit-length edges.)

4) a) Let $k \geqslant 1$. Give an isometric embedding of ℓ_1^k to $\ell_\infty^{2^k}$.

   b) Devise an algorithm that, given a set X of n points in $\mathbb{R}^k$, computes the diameter of X under the ℓ_1 norm using $O(k2^k n)$ arithmetic operations.

   c) Can you reduce the number of arithmetic operations to $O(2^k n)$ (or even further)?

5) Consider the set of integers $V = \{4, 5, 9\}$ and let $d(u, v) = |u - v|$ for $u, v \in V$. Write d as a conic combination of cut metrics

$$d = \sum_{S \in 2^S} \mu_S \cdot d_S,$$

   where $\mu_S \in \mathbb{R}_{\geqslant 0}$ and d_S is the cut-metric induced by S.

6) Let $R, B \subseteq \mathbb{R}^2$ be two finite sets of n distinct points. We consider the usual Euclidean metric. Show the inequality

$$\sum_{u,v \in R} d(u, v) + \sum_{u,v \in B} d(u, v) \leqslant \sum_{u \in R} \sum_{v \in B} d(u, v).$$