Problem 1. In the All-Pairs LCA in DAGs problem one is given a directed acyclic graph $G = (V, E)$, and the goal is to find for every pair of vertices $u, v \in V$ a vertex that is their least common ancestor, or decide that there is no such vertex.

Vertex $w$ is called a common ancestor of $u$ and $v$ if there are directed paths from $w$ to $u$ and from $w$ to $v$. It is a least common ancestor if no other common ancestor is reachable from $w$. Note that there can be many least common ancestors of a pair of vertices.

Prove that there is no truly subcubic combinatorial algorithm for All-Pairs LCA in DAGs unless BMM Hypothesis is false.

Problem 2. Given two $n \times n$ Boolean matrices $A$ and $B$, the Minimum Witness problem asks to compute the $n \times n$ integer matrix $C$ such that

$$C[i][j] = \min\{k \mid A[i][k] \land B[k][j]\}.$$

Design a truly subcubic algorithm for Minimum Witness.

Hint: For a carefully chosen parameter $\mu$, split $A$ into $n^{1-\mu}$ matrices of size $n \times n^\mu$ and $B$ into $n^{1-\mu}$ matrices of size $n^\mu \times n$.

Problem 3. Give a truly subcubic algorithm for All-Pairs LCA in DAGs. Hint: Reduce it to Minimum Witness.

Problem 4. Prove that the following problem does not admit a truly subquadratic algorithm unless 3SUM does: Given a set of points in the plane with integer coordinates, is there a line that contains at least three of these points?

Hint: What is the sum of roots of polynomial $x^3 + ax + b$, for any $a$ and $b$?

Problem 5. The $k$-SUM problem is given $k$ sets of $n$ integers each determine if there is a choice of one element from each set such that the elements sum up to 0. Prove that ETH implies that $k$-SUM cannot be solved in $n^{o(k)}$ time.

Problem 6. Let $k \equiv 0 \mod 3$. Find an algorithm that given an undirected graph on $n$ vertices determines if the graph contains a $k$-clique in time $O(n^{\frac{2}{3}}k)$.

This problem set adds 1 point to the threshold for grade 4.0, and 3 points for 6.0