Problem 1. In the All-Pairs LCA in DAGs problem one is given a directed acyclic graph $G = (V, E)$, and the goal is to find for every pair of vertices $u, v \in V$ a vertex that is their *least common ancestor*, or decide that there is no such vertex.

Vertex w is called a *common ancestor* of u and v if there are directed paths from w to u and from w to v. It is a *least common ancestor* if no other common ancestor is reachable from w. Note that there can be many least common ancestors of a pair of vertices.

Prove that there is no truly subcubic combinatorial algorithm for All-Pairs LCA in DAGs unless BMM Hypothesis is false.

Problem 2. Given two $n \times n$ Boolean matrices A and B, the Minimum Witness problem asks to compute the $n \times n$ integer matrix C such that

$$C[i][j] = \min\{k \mid A[i][k] \land B[k][j]\}.$$

Design a truly subcubic algorithm for Minimum Witness.

Hint: For a carefully chosen parameter μ, split A into $n^{1-\mu}$ matrices of size $n \times n^{\mu}$ and B into $n^{1-\mu}$ matrices of size $n^{\mu} \times n$.

Problem 3. Give a truly subcubic algorithm for All-Pairs LCA in DAGs. Hint: Reduce it to Minimum Witness.

Problem 4. Prove that the following problem does not admit a truly subquadratic algorithm unless 3SUM does: Given a set of points in the plane with integer coordinates, is there a line that contains at least three of these points?

Hint: What is the sum of roots of polynomial $x^3 + ax + b$, for any a and b?

Problem 5. The k-SUM problem is given k sets of n integers each determine if there is a choice of one element from each set such that the elements sum up to 0. Prove that ETH implies that k-SUM cannot be solved in $n^{o(k)}$ time.

Problem 6. Let $k \equiv 0 \mod 3$. Find an algorithm that given an undirected graph on n vertices determines if the graph contains a k-clique in time $\mathcal{O}(n^{\frac{\omega}{3}k})$.

This problem set adds ***1 point*** *to the threshold for grade 4.0, and* ***3 points*** *for 6.0*