Problem 1. Prove that the Negative Triangle problem admits a truly subcubic algorithm if and only if the Negative Triangle problem restricted to tripartite graphs admits a truly subcubic algorithm.

Problem 2. In the Metricity problem we are given an $n \times n$ matrix A with positive integer entries, and we have to determine whether for all $i, j, k \in [n]$ we have $A[i][j] \leqslant A[i][k] + A[k][j]$. Prove that Metricity is subcubic equivalent to APSP.

Problem 3. Prove that BMM Hypothesis is equivalent to the following hypothesis: Given an n-node undirected graph, no combinatorial algorithm can detect if the graph contains a triangle in time $\mathcal{O}(n^{3-\varepsilon})$, for any $\varepsilon > 0$.

Problem 4. The Hitting Set Hypothesis says that, given sets $S_1, \ldots, S_n, T_1, \ldots, T_n \subseteq [d]$, no algorithm can determine in $\mathcal{O}(n^{2-\varepsilon}\text{poly}(d)$ time whether there is a set S_i that intersect every set T_j. Prove that Hitting Set Hypothesis implies OV Hypothesis.

Problem 5. Design an $\mathcal{O}(n^{2.99})$ time randomized algorithm for APSP in directed unweighted graphs. Hint: Consider short and long paths separately. For short paths, iteratively multiply the adjacency matrix. For long paths, sample a subset of vertices that hits all such paths with high probability, and compute shortest paths to and from that subset.

Problem 6. Prove that if $(\min, +)$-product can be solved in $T(n)$ time, then APSP can be solved in $\mathcal{O}(T(n))$ time.

Hint: `https://people.csail.mit.edu/virgi/6.s078/hw2.pdf`

*This problem set adds **2 points** to the threshold for grade 4.0, and **4 points** for 6.0*