Problem 1. Prove that if the Variable-Deletion-2-SAT problem is solvable in $O^*(f(k))$, then the Almost-2-SAT problem is also solvable in $O^*(f(k))$ time. Note that the reduction has to preserve (the asymptotics of) the function $f$ but can alter polynomial factors hidden in the $O^*$ notation.

Problem 2. In the Cluster Vertex Deletion problem we are given an undirected graph and an integer $k$, and the task is to decide if there exists a set of at most $k$ vertices whose removal makes the graph a cluster graph (i.e. each connected component is a clique). Solve Cluster Vertex Deletion in $O^*(2^k)$ time. Hint: prove that the Disjoint Cluster Vertex Deletion can be solved in polynomial time, and use the iterative compression technique.

Problem 3. Prove that if a graph has minimum degree $d$ then it has treewidth at least $d$.

Problem 4. Prove that every clique of a graph has to be contained in some bag of its tree decomposition.

Problem 5. A coloring number (also called degeneracy number) of a graph is the smallest integer $\delta$ such that the vertices of the graph can be ordered in such a way that each vertex has at most $\delta$ neighbors appearing before it in the order. Prove that graphs of treewidth $k$ have coloring number at most $k$.

Problem 6. A graph is outerplanar if it can be embedded in the plane in such a way that all vertices are on the outer face. Prove that outerplanar graphs have treewidth at most 2.

Definition. A graph is a $k$-tree if it can be obtained from the following process. We start with a clique on $k$ vertices. Iteratively, we add a vertex to the graph and connect it with some $k$ existing vertices which form a clique.

Problem 7. Prove that a $k$-tree has treewidth exactly $k$.

Problem 8. Prove that every graph of treewidth $k$ is a subgraph of some $k$-tree.

Problem 9. Propose a $2^{O(w)} n$ time algorithm for the Vertex Cover problem in graphs of treewidth at most $w$. Hint: for $t \in T$ and $X \subset B_t$ let $dp[t][X]$ denote the size of the smallest vertex cover of the subgraph induced by vertices appearing in bags below $t$ (let us root the tree decomposition at an arbitrary node, so that “below” is well defined) such that the vertex cover contains $X$.

This problem set adds 2-points 1 point to the threshold for grade 4.0, and 6-points 5 points for 6.0. In the initial version of this problem set there was a serious mistake in Problem 5 (at least instead of at most), therefore both thresholds are now decreased.