Problem 1. Prove that if the Variable-Deletion-2-SAT problem is solvable in $\mathcal{O}^*(f(k))$, then the Almost-2-SAT problem is also solvable in $\mathcal{O}^*(f(k))$ time. Note that the reduction has to preserve (the asymptotics of) the function f but can alter polynomial factors hidden in the $\mathcal{O}^*$ notation.

Problem 2. In the Cluster Vertex Deletion problem we are given an undirected graph and an integer k, and the task is to decide if there exists a set of at most k vertices whose removal makes the graph a cluster graph (i.e. each connected component is a clique). Solve Cluster Vertex Deletion in $\mathcal{O}^*(2^k)$ time. Hint: prove that the Disjoint Cluster Vertex Deletion can be solved in polynomial time, and use the iterative compression technique.

Problem 3. Prove that if a graph has minimum degree d then it has treewidth at least d.

Problem 4. Prove that every clique of a graph has to be contained in some bag of its tree decomposition.

Problem 5. A *coloring number* (also called *degeneracy number*) of a graph is the smallest integer δ such that the vertices of the graph can be ordered in such a way that each vertex has at most δ neighbors appearing before it in the order. Prove that graphs of treewidth k have coloring number at most k.

Problem 6. A graph is *outerplanar* if it can be embedded in the plane in such a way that all vertices are on the outer face. Prove that outerplanar graphs have treewidth at most 2.

Definition. A graph is a k-*tree* if it can be obtained from the following process. We start with a clique on k vertices. Iteratively, we add a vertex to the graph and connect it with some k existing vertices which form a clique.

Problem 7. Prove that a k-tree has treewidth exactly k.

Problem 8. Prove that every graph of treewidth k is a subgraph of some k-tree.

Problem 9. Propose a $2^{\mathcal{O}(w)}n$ time algorithm for the Vertex Cover problem in graphs of treewidth at most w. Hint: for $t \in T$ and $X \subset B_t$ let $dp[t][X]$ denote the size of the smallest vertex cover of the subgraph induced by vertices appearing in bags below t (let us root the tree decomposition at an arbitrary node, so that "below" is well defined) such that the vertex cover contains X.

*This problem set adds ~~**2 points**~~ **1 point** to the threshold for grade 4.0, and ~~**6 points**~~ **5 points** for 6.0. In the initial version of this problem set there was a serious mistake in Problem 5 (at least instead of at most), therefore both thresholds are now decreased.*