Definition. A *tournament* is a directed graph $G = (V, E)$ such that for each pair of distinct nodes $u, v \in V$, $u \neq v$, either $(u, v) \in E$ or $(v, u) \in E$. In other words, we start with an undirected clique, and we select for each edge one of its two possible orientations.

Definition. In the Feedback Arc Set in Tournaments problem, we are given a tournament and an integer k, and we have to remove at most k edges to make the graph acylic.

Problem 1. Let T be a tournament. Show that the three conditions are equivalent: (a) T has no cycle; (b) T has no triangle; and (c) T has exactly one topological ordering.

Problem 2. Let G be a directed graph that can be obtained from a tournament by removing at most ℓ edges. Prove that if G has a cycle, it has a cycle of length at most $\mathcal{O}(\sqrt{\ell})$. Use this observation to design an FPT algorithm for Feedback Arc Set in Tournaments.

Problem 3. Show an $\mathcal{O}^*(3^k)$ time algorithm for Feedback Arc Set in Tournaments. Hint: think of reversing edges instead of removing them.

Definition. In the Feedback Vertex Set (FVS) problem, we are given an undirected graph and an integer k, and we have to remove at most k vertices to make the graph acylic, i.e. a forest.

Problem 4. Show that an undirected n-node graph of minimum degree 3 always has a cycle of length at most $\mathcal{O}(\log n)$. Show a $(\log n)^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$ time algorithm for FVS.

Problem 5. Given an undirected graph G and an integer k, the Triangle Packing problem asks to find k vertex-disjoint triangles in G. Design a randomized $\mathcal{O}^*(c^k)$ time algorithm for Triangle Packing, for a constant c.

Problem 6. Propose a randomized $\mathcal{O}^*(c^k)$ time algorithm for the problem of deciding whether an undirected graph has a cycle of length at least k. Hint: show that if a graph contains a cycle of length at least $2k$, then, we can contract an arbitrary edge, and the graph will still contain a cycle of length at least k.

*This problem set adds **2 points** to the threshold for grade 4.0, and **4 points** for 6.0*