Graph theory - solutions to problem set 12

1. Calculate the eigenvalues and eigenvectors of the adjacency matrix of C_4.

   Solution: The adjacency matrix of C_4 is $\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$. As C_4 is 2-regular, we know that $(1,1,1,1)$ is an eigenvector with eigenvalue 2. One can guess (e.g. from problem 5, noting that C_4 is bipartite) that $(1,1,-1,-1)$ is another eigenvector, with eigenvalue -2. Also, the matrix has rank 2 (and hence 2-dimensional null-space), so it will have 0 as an eigenvalue with a 2-dimensional eigenspace. This eigenspace contains all vectors of the form $(x,-x,y,-y)$.

2. (a) Let G be a graph, and let k be a positive integer. Prove that for every $x, y \in V(G)$, $A_G^k(x,y)$ is equal to the number of walks in G of length k with endpoints x and y.

   Solution: We can prove this by induction on k. For $k=1$, the statement is trivial. Suppose the statement is true for k. We will show that it is true for $k+1$. The number of walks in G of length $k+1$ with endpoints x and y is equal to the sum of numbers of walks in G of length k with end points x and z where the sum is taken over all the neighbours z of y. By induction hypothesis, this is equal to $\sum_z A_G^k(x,z)A_G(z,y) = A_G^{k+1}(x,y)$, as required.

   (b) Let G be a graph on n vertices and let $\lambda_1, ..., \lambda_n$ be all the eigenvalues of A_G. Show that

   $$\sum_{i=1}^{n} \lambda_i^2 = 2|E(G)|.$$

   Solution: By (a), for any $x \in V(G)$, $A_G^2(x,x)$ is equal to the number of closed walks of length 2 from x to itself. It is easy to see that $\sum_{x\in V(G)} A_G^2(x,x)$ is exactly $2|E(G)|$. To finish the proof, one just needs to recall from linear algebra that $\sum_{i=1}^n \lambda_i^2 = Tr(A_G^2) = \sum_{x\in V(G)} A_G^2(x,x)$.

   (Indeed, if λ is an eigenvalue of A_G then λ^2 is an eigenvalue of A_G^2 with the same multiplicity. Diagonalizing A_G^2 as $S^{-1}A_G^2S = D$, where D is the diagonal matrix with the eigenvalues of A_G^2 on the diagonal, gives $Tr(A_G^2) = Tr(S^{-1}A_G^2S) = Tr(D) = \sum_{i=1}^n \lambda_i^2$, as required.)

3. Let G be a graph that is $\text{sgr}(n,d,\lambda,\mu)$. Calculate n as a function of d, λ and μ.

   Solution: Let us count cherries as in the lecture. Counting from the root, there are $\binom{d}{2}$ cherries at every vertex, so there are $n\binom{d}{2}$ cherries in total. Counting from the pairs, there are exactly λ cherries sitting on every edge, and there are μ cherries sitting on every non-edge. So if m denotes the number of edges, then the number of cherries is $\lambda m + \mu(\binom{n}{2} - m)$. Plugging in $m = nd/2$, we get

   $$\frac{nd(d-1)}{2} = \frac{\lambda nd}{2} + \frac{\mu n(n-d-1)}{2}.$$

   Rearranging, this gives

   $$\mu n^2 + n((\lambda+1)d - \mu(d+1) - d^2) = 0$$

   and hence (using $n \neq 0$)

   $$n = \frac{d^2 + \mu(d+1) - (\lambda+1)d}{\mu}.$$

4. Let G be a d-regular graph. Prove that if λ is an eigenvalue of A_G, then $|\lambda| \leq d$.

   Solution: Let v be an eigenvector of A_G with eigenvalue λ, and suppose its i'th coordinate v_i is the largest in absolute value. We know, by definition, that the i'th coordinate of $A_G v$ is λv_i. On the other hand, this coordinate is equal to the product of the i'th row of A_G and v. As G is d-regular, the i'th row contains d 1-entries, say at coordinates J. But then

   $$|\lambda v_i| = |(A_G v)_i| = |\sum_{j \in J} v_j| \leq \sum_{j \in J} |v_j| \leq d|v_i|,$$

   and hence $|\lambda| \leq d$.

5. Let G be a bipartite graph. Prove that if λ is an eigenvalue of A_G, then $-\lambda$ is also an eigenvalue.

   Solution: Suppose G is bipartite, with parts S and T of sizes s and t. The idea is that for some eigenvector, one can "flip" the sign of the coordinates corresponding to one part and get another eigenvector with opposite eigenvalue.

   Let us see how this works in detail: For some $s \times t$ matrix B we have

   $$A_G = \begin{bmatrix} O_{s\times s} & B \\ B^T & O_{t\times t} \end{bmatrix}.$$

   Since λ is an eigenvalue, we have

   $$\begin{bmatrix} \lambda v \\ \lambda w \end{bmatrix} = \lambda \begin{bmatrix} v \\ w \end{bmatrix} = A_G \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} Bw \\ B^T v \end{bmatrix}.$$

   So $Bw = \lambda v$ and $B^T v = \lambda w$. But then

   $$A_G \begin{bmatrix} v \\ -w \end{bmatrix} = \begin{bmatrix} -Bw \\ B^T v \end{bmatrix} = -\lambda \begin{bmatrix} v \\ -w \end{bmatrix}.$$

   Thus $-\lambda$ is also an eigenvalue.

6. Let G be a graph and let p be the number of positive eigenvalues of A_G (with multiplicity), and let n be the number of negative eigenvalues of A_G (with multiplicity). Prove that the edge set of G cannot be partitioned into fewer than $\max(p, n)$ complete bipartite graphs.

   Solution: The idea is similar to the case of complete graphs seen in the lecture. The difference is the way we bound $v^T A_G v$ away from 0 instead of (5) in the proof. So suppose G can be partitioned into complete bipartite graphs $G[X_i, Y_i]$, and let W be the p-dimensional subspace spanned by the eigenvectors of A_G with positive eigenvalues. It is well-known from linear algebra that $v^T A_G v > 0$ for every nonzero $v \in W$. On the other hand, any vector $v \in W$ such that $\sum_{x \in X_i} v(x) = 0$ for every i will satisfy $v^T A_G v = 0$ (see the original proof). If G can be partitioned into fewer than p complete bipartite graphs, then these are fewer than p linear equations, so there is a nonzero such $v \in W$, a contradiction.

   We similarly get a contradiction for negative eigenvalues (there $v^T A_G v < 0$ if v is in the n-dimensional subspace induced by negative eigenvectors).

7. Let G be a graph that is $\mathrm{sgr}(n, d, \lambda, \mu)$. Calculate the eigenvalues of A_G as a function of n, d, λ, μ.

   Solution: We repeat the argument from the lecture. A_G^2 has d in the diagonal, λ at the entries of the edges and μ everywhere else, so $A_G^2 + (\mu - \lambda)A_G - (d - \mu)I = \mu J$ (where J is the all-1 matrix). As G is d-regular, we know that $v = (1, \ldots, 1)$ is an eigenvector of A_G with eigenvalue d. As A_G is symmetric, it has an orthogonal basis of eigenvectors extending v. Let u be any other vector in this basis, say with eigenvalue c. Then $\langle v, u\rangle = 0$, so $Ju = 0$. But then

   $$(A_G^2 + (\mu - \lambda)A_G - (d - \mu)I)u = c^2 u + (\mu - \lambda)cu - (d - \mu)u = 0,$$

so $c^2 + (\mu - \lambda)c - (d - \mu) = 0$. Solving this equation, we get that all eigenvalues other than d are of the form

$$c_{1,2} = \frac{\lambda - \mu \pm \sqrt{(\mu - \lambda)^2 + 4(d - \mu)}}{2}.$$

To complete the solution, one would also need to check that both of both of these eigenvalues occur. This is not really the point of the exercise, so we will skip the details.