Graph theory - solutions to problem set 11

1. Let V be a set of n vertices. We will construct a random graph G with m edges. For each $e \in V \times V$ we perform a random experiment, the outcome of which will determine if e is and edge of G. The experiments are performed independently, and for every one of them the probability of success is p.

   What is the probability that we obtain some fixed graph $G_0 = (V, E_0)$, where $|E_0| = m$?

   Solution: The probability is $p^m q^{\binom{n}{2}-m}$, where $q = 1 - p$.

2. Let $(\Omega, \mathbb{P})$ be a probability space. Prove that for any collection of events $E_1, \ldots, E_k$, we have

$$\mathbb{P}\left[\bigcup_{i=1}^{k} E_i\right] \leq \sum_{i=1}^{k} \mathbb{P}[E_i],$$

   and if $E_1, \ldots, E_k$ are disjoint events, then we have equality here.

   Solution: The proof is by induction on n. For two events E_1 and E_2 we have that

$$\mathbb{P}[E_1 \cup E_2] = \sum_{\omega \in E_1 \cup E_2} p(\omega) \leq \sum_{\omega \in E_1} p(\omega) + \sum_{\omega \in E_2} p(\omega) = \mathbb{P}[E_1] + \mathbb{P}[E_2].$$

   Assume that the statement holds for $n - 1$ and let us prove it for n:

$$\mathbb{P}\left[\bigcup_{1}^{n} E_i\right] = \mathbb{P}\left[\left(\bigcup_{1}^{n-1} E_i\right) \cup E_n\right] \leqslant \mathbb{P}\left[\bigcup_{1}^{n-1} E_i\right] + \mathbb{P}[E_n] \leqslant \sum_{1}^{n-1} \mathbb{P}[E_i] + \mathbb{P}[E_n] = \sum_{1}^{n} \mathbb{P}[E_i].$$

3. Let σ be an arbitrary permutation of $\{1, \ldots, n\}$, selected uniformly at random from the set of all permutations (that is, each permutation is selected with probability $\frac{1}{n!}$). What is the expectation of the number of fixed points in σ? (Recall that i is a fixed point if $\sigma(i) = i$.)

   Solution: Let X_i be a random variable being 1 if the i-th position is a fixed point, and 0 otherwise. Then, by linearity of expectation, the expected number of fixed points in a random permutation are simply $\sum_{i=1}^{n} \mathbb{E}[X_i]$. On the other hand, $\mathbb{E}[X_i] = 1/n$, so we obtain that

$$\sum_{i=1}^{n} \mathbb{E}[X_i] = n \cdot 1/n = 1.$$

4. Take a complete graph K_n where each edge is independently colored red, green or blue with probability 1/3. What is the expected number of red cliques of size a in this graph?

   Solution: We assume that vertices of G are labeled from 1 to n. Let X be the number of crossing edges. We decompose

$$X = \sum_{I \in \binom{[n]}{a}} X_I,$$

   where $\binom{[n]}{a}$ is the set of a-element subsets of the set $\{1, \ldots, n\}$ and X_I is the indicator random variable for clique on vertices with indices from I being red colored.

   By linearity of expectation, we have

$$\mathbb{E}[X] = \sum_{I \in \binom{[n]}{a}} \mathbb{E}[X_I].$$

   On the other hand $\mathbb{E}[X_I] = \left(\frac{1}{3}\right)^{\binom{a}{2}}$, and, finally, the expected number of red cliques of size a in this graph is

$$\mathbb{E}[X] = \binom{n}{a} \left(\frac{1}{3}\right)^{\binom{a}{2}}$$

5. Let G be a graph with m edges, and let $X \subseteq V(G)$ be a random set that contains each vertex of G independently with probability $1/2$. What is the expected number of edges in the induced subgraph $G[X]$?
(Here $G[X]$ is the subgraph of G with vertex set X, and contains all edges in G with both ends in X.)

Solution: Let Y be the number of edges in $G[X]$. Then $Y = \sum_{e \in E(G)} Y_e$, where Y_e is the indicator of the event that the edge e is induced by X. This event occurs if and only if both endpoints of e are included in X, which has probability $1/4$. So by the linearity of expectation, $\mathbb{E}[Y] = \sum_{e \in E(G)} \mathbb{E}[Y_e] = m/4$.

6. Let G be a graph with m edges, and let k be a positive integer. Prove that the vertices of G can be colored with k colors in such a way that there are at most m/k monochromatic edges (i.e., edges with both endpoints colored the same).

Solution: We assume that edges of G are labeled from 1 to m. For any $1 \leqslant i \leqslant m$, we define the random variable X_i being 1 whenever endpoints of the i-th edge are colored by the same color, and 0 otherwise. Let $X = \sum_i^m X_i$. Then $\mathbb{E}[X] = \sum_i^m \mathbb{E}[X_i]$. On the other hand, it is easy to see that $\mathbb{E}[X_i] = \frac{k}{k^2} = \frac{1}{k}$. Therefore, $\mathbb{E}[X] = \frac{m}{k}$. Thus, there is a coloring of the vertices of G with k colors such that at most $\frac{m}{k}$ edges of G connect two vertices with the same color.

7. Prove that if G has $2n$ vertices and e edges then it contains a bipartite subgraph with at least $e\frac{n}{2n-1}$ edges. [Use a random partition of the vertices into two parts of size n]

Solution: The proof is similar to the probabilistic proof for the bound $e/2$ given in lectures, but now we choose a more subtle probability space. Let A be an n-element subset of $V(G)$ chosen uniformly among all n-element subsets of $V(G)$. Set $B = V(G) - T$. Call an edge $\{x, y\}$ crossing if exactly one of x, y are in A. Then any edge $\{x, y\}$ now has probability $\frac{n}{2n-1}$ of being crossing.

We complete the proof with the standard arguments. Let X be the number of crossing edges and X_{xy} be the indicator random variable for $\{x, y\}$ being crossing. By linearity of expectation,

$$\mathbb{E}[X] = \sum_{\{x,y\} \in E(G)} \mathbb{E}[X_{xy}] = e\frac{n}{2n-1}.$$

Thus, there is a choice of T such that $X \geqslant e\frac{n}{2n-1}$ and the set of those crossing edges form a bipartite graph.