Graph theory - problem set 12

December 5, 2019

Exercises

1. Let G be a graph on 6 vertices such that $\alpha(G) < 3$. Prove that G contains a triangle.

2. Using k colors, construct a coloring of the edges of the complete graph on 2^k vertices without creating a monochromatic triangle.

3. The lower bound for $R(s,s)$ that we saw in the lecture is not a constructive proof: it merely shows the *existence* of a red-blue coloring not containing any monochromatic copy of K_p by bounding the number of bad graphs. Give an explicit coloring on $K_{(s-1)^2}$ that proves $R(s,s) > (s-1)^2$.

4. A random graph $G(n,p)$ is a probability space of all labeled graphs on n vertices $\{1,2,\dots,n\}$, where for each pair $1 \leq i < j \leq n$, (i,j) is an edge of $G(n,p)$ with probability p, independently of any other edge (you can think of a sequence of independent coin tosses for each edge). Compute the following:
   (a) the expected number of edges in $G(n,p)$;
   (b) the expected degree of a vertex in $G(n,p)$;
   (c) the expected number of triangles (cycles of length 3) in $G(n,p)$;
   (d) the expected number of paths of length 2 in $G(n,p)$;
   (e) the probability that the degree of a given vertex v is exactly k.

5. Prove that $R(n_1,\dots,n_k) \leq R(n_1,\dots,n_{k-2},R(n_{k-1},n_k))$. Deduce that for every k and n, there is an N such that any k-coloring of the edges of K_N contains a monochromatic K_n.

6. Show that the edges of K_n can be colored with 3 colors so that the number of monochromatic triangles is at most $\frac{1}{9}\binom{n}{3}$.

7. (a) Show that if for some real number $0 \leq p \leq 1$ we have $\binom{n}{s}p^{\binom{s}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}} < 1$, then $R(s,t) > n$.
   (b) Deduce that there is a positive constant c such that $R(4,t) \geq c \cdot \frac{t^{3/2}}{\log^{3/2} t}$.

   [Hint: Use $p = n^{-2/3}$ in (a) to deduce (b).]

8. Prove that for every $k \geq 2$ there exists an integer N such that every coloring of $[N] = \{1,\dots,N\}$ with k colors contains three numbers a,b,c satisfying $ab = c$ that have the same color.

9. (a) Prove that $R(4,3) \leq 10$, i.e., any graph on 10 vertices contains a clique of size 4 or an independent set of size 3.
   (b) Prove that $R(4,3) \leq 9$.