Graph theory - problem set 12
December 5, 2019

Exercises

1. Let $G$ be a graph on 6 vertices such that $\alpha(G) < 3$. Prove that $G$ contains a triangle.

2. Using $k$ colors, construct a coloring of the edges of the complete graph on $2^k$ vertices without creating a monochromatic triangle.

3. The lower bound for $R(s, s)$ that we saw in the lecture is not a constructive proof: it merely shows the existence of a red-blue coloring not containing any monochromatic copy of $K_p$ by bounding the number of bad graphs. Give an explicit coloring on $K_{(s-1)^2}$ that proves $R(s, s) > (s - 1)^2$.

4. A random graph $G(n, p)$ is a probability space of all labeled graphs on $n$ vertices $\{1, 2, \ldots, n\}$, where for each pair $1 \leq i < j \leq n$, $(i, j)$ is an edge of $G(n, p)$ with probability $p$, independently of any other edge (you can think of a sequence of independent coin tosses for each edge). Compute the following:
   (a) the expected number of edges in $G(n, p)$;
   (b) the expected degree of a vertex in $G(n, p)$;
   (c) the expected number of triangles (cycles of length 3) in $G(n, p)$;
   (d) the expected number of paths of length 2 in $G(n, p)$;
   (e) the probability that the degree of a given vertex $v$ is exactly $k$.

5. Prove that $R(n_1, \ldots, n_k) \leq R(n_1, \ldots, n_{k-2}, R(n_{k-1}, n_k))$. Deduce that for every $k$ and $n$, there is an $N$ such that any $k$-coloring of the edges of $K_N$ contains a monochromatic $K_n$.

6. Show that the edges of $K_n$ can be colored with 3 colors so that the number of monochromatic triangles is at most $\frac{1}{2} \binom{n}{3}$.

7. (a) Show that if for some real number $0 \leq p \leq 1$ we have $\binom{n}{2}p^{(2)} + \binom{n}{1}(1-p)\binom{1}{1} < 1$, then $R(s, t) > n$. 
   (b) Deduce that there is a positive constant $c$ such that $R(4, t) \geq c \cdot \frac{3^{3/2}}{\log^{1/2} t}$.

   [Hint: Use $p = n^{-2/3}$ in (a) to deduce (b).]

8. Prove that for every $k \geq 2$ there exists an integer $N$ such that every coloring of $[N] = \{1, \ldots, N\}$ with $k$ colors contains three numbers $a, b, c$ satisfying $ab = c$ that have the same color.

9. (a) Prove that $R(4, 3) \leq 10$, i.e., any graph on 10 vertices contains a clique of size 4 or an independent set of size 3.
   (b) Prove that $R(4, 3) \leq 9$. 