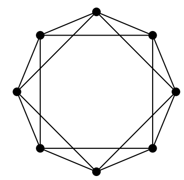
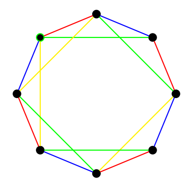
Graph theory - problem set 9

November 14, 2019

1. Determine the edge-chromatic number of the graph below.



Solution:



2. (a) Find the edge-chromatic number of K_{2n+1} (don't use Vizing's theorem).

   (b) Find the edge-chromatic number of K_{2n}.

   Solution: $\chi'(K_{2k-1}) = 2k-1$: To get a $2k-1$-coloring, place the vertices v_i on a circle with equal spacing. Then for each vertex v_i, give the same color to the edges $v_{i-1}v_{i+1}$, $v_{i-2}v_{i+2}$, etc. (these edges will be parallel). This way we color all the edges with $2k-1$ colors.

   Suppose we could color the edges of K_{2k-1} with $2k-2$ colors. Each color class has at most $k-1$ edges, so with $2k-2$ colors we can color at most $(2k-2)(k-1)$ edges. But K_{2k-1} has $\binom{2k-1}{2} = (2k-1)(k-1)$ edges, so this can't work.

   $\chi'(K_{2k}) = 2k-1$: Now place $2k-1$ of the vertices v_i on a circle with equal spacing, and put the remaining vertex u at the center of the circle. Then for each v_i, color in the same way as in the odd case, and also give that color to the edge uv_i. This gives an edge coloring with $2k-1$ colors.

   In this case we have $\Delta(K_{2k}) = 2k-1$, so by a theorem from class there is no edge coloring with fewer colors.

3. Let G be a 3-regular graph with $\chi'(G) = 4$. Prove that G does not have a Hamilton cycle.

   Solution: Note that since $\sum d(v) = 2|E(G)|$ and every $d(v) = 3$, G must have an even number of vertices. Suppose that G has a Hamilton cycle C. It must be even, so we can color it with 2 colors. Every vertex has 2 edges from C and one other edge, so the edges not in C form a matching. Hence we can color these edges with one color. This gives an edge-coloring of G with 3 colors, contradicting $\chi'(G) = 4$.

4. Prove that every bipartite graph satisfies $\chi'(G) = \Delta(G)$.

   We apply induction on m, the number of edges of G. If $n = 0$ the assertion holds trivially. Now assume $m > 1$, and that the assertion holds for graphs with fewer edges. Let $\Delta := \Delta G$, pick an edge $xy \in G$, and choose a Δ-edge-colouring of $G - xy$ using the induction hypothesis. Let us refer to the edges coloured α as *α-edges*,etc.

   In $G - xy$, each of x and y is incident with at most $\Delta - 1$ edges. Hence, there are $\alpha, \beta \in \{1, \ldots, \Delta\}$ such that x is not incident with an α-edge and y is not incident with a β-edge. If $\alpha = \beta$, we can colour the edge xy with this colour and are done; so we may assume that $\alpha \neq \beta$, and that x is incident with a β-edge.

Let us extend this edge to a a maximal walk W from x whose edges are coloured β and α alternately. Since no such walk contains a vertex twice, W exists and is a path. Moreover, W does not contain y: if it did, it would end in y on an α-edge (by choice of β) and thus have even length, so $W + xy$ would be an odd cycle in G. We now recolour all the edges on W, swapping α with β. By the choice of α and the maximality of W, adjacent edges of $G - xy$ are still coloured differently. We have thus found a Δ-edge-colouring of $G - xy$ in which neither x nor y is incident with a β-edge. Colouring xy with β, we extend this colouring to a Δ-edge-colouring of G.