1. Determine the edge-chromatic number of the graph below.

Solution:

2. (a) Find the edge-chromatic number of $K_{2n+1}$ (don't use Vizing’s theorem).
   (b) Find the edge-chromatic number of $K_{2n}$.

   **Solution:** $\chi'(K_{2k-1}) = 2k-1$: To get a $2k-1$-coloring, place the vertices $v_i$ on a circle with equal spacing. Then for each vertex $v_i$, give the same color to the edges $v_{i-1}v_{i+1}, v_{i-2}v_{i+2}$, etc. (these edges will be parallel). This way we color all the edges with $2k-1$ colors.

   Suppose we could color the edges of $K_{2k-1}$ with $2k-2$ colors. Each color class has at most $k-1$ edges, so with $2k-2$ colors we can color at most $(2k-2)(k-1)$ edges. But $K_{2k-1}$ has $(\binom{2k-1}{2}) = (2k-1)(k-1)$ edges, so this can’t work.

   $\chi'(K_{2k}) = 2k - 1$: Now place $2k-1$ of the vertices $v_i$ on a circle with equal spacing, and put the remaining vertex $u$ at the center of the circle. Then for each $v_i$, color in the same way as in the odd case, and also give that color to the edge $uv_i$. This gives an edge-coloring with $2k-1$ colors.

   In this case we have $\Delta(K_{2k}) = 2k-1$, so by a theorem from class there is no edge coloring with fewer colors.

3. Let $G$ be a 3-regular graph with $\chi'(G) = 4$. Prove that $G$ does not have a Hamilton cycle.

   **Solution:** Note that since $\sum d(v) = 2|E(G)|$ and every $d(v) = 3$, $G$ must have an even number of vertices. Suppose that $G$ has a Hamilton cycle $C$. It must be even, so we can color it with 2 colors. Every vertex has 2 edges from $C$ and one other edge, so the edges not in $C$ form a matching. Hence we can color these edges with one color. This gives an edge-coloring of $G$ with 3 colors, contradicting $\chi'(G) = 4$.

4. Prove that every bipartite graph satisfies $\chi'(G) = \Delta(G)$.

   We apply induction on $m$, the number of edges of $G$. If $n = 0$ the assertion holds trivially. Now assume $m > 1$, and that the assertion holds for graphs with fewer edges. Let $\Delta := \Delta G$, pick an edge $xy \in G$, and choose a $\Delta$-edge-colouring of $G - xy$ using the induction hypothesis. Let us refer to the edges coloured $\alpha$ as $\alpha$-edges, etc.

   In $G - xy$, each of $x$ and $y$ is incident with at most $\Delta - 1$ edges. Hence, there are $\alpha, \beta \in \{1, \ldots, \Delta\}$ such that $x$ is not incident with an $\alpha$-edge and $y$ is not incident with a $\beta$-edge. If $\alpha = \beta$, we can colour the edge $xy$ with this colour and are done; so we may assume that $\alpha \neq \beta$, and that $x$ is incident with a $\beta$-edge.
Let us extend this edge to a maximal walk $W$ from $x$ whose edges are coloured $\beta$ and $\alpha$ alternately. Since no such walk contains a vertex twice, $W$ exists and is a path. Moreover, $W$ does not contain $y$: if it did, it would end in $y$ on an $\alpha$-edge (by choice of $\beta$) and thus have even length, so $W + xy$ would be an odd cycle in $G$. We now recolour all the edges on $W$, swapping $\alpha$ with $\beta$. By the choice of $\alpha$ and the maximality of $W$, adjacent edges of $G - xy$ are still coloured differently. We have thus found a $\Delta$-edge-colouring of $G - xy$ in which neither $x$ nor $y$ is incident with a $\beta$-edge. Colouring $xy$ with $\beta$, we extend this colouring to a $\Delta$-edge-colouring of $G$. 