Graph theory - solutions to problem set 8

Exercises

1. You are given an algorithm testing if a graph G has a perfect matching or not. If G has a perfect matching, it always outputs *yes*. Otherwise, it outputs *yes* only with probability less than $\frac{1}{2}$.
   Given $\epsilon > 0$, use this algorithm to design an algorithm which always outputs *yes* if the graph has a perfect matching and if not, it outputs *yes* only with probability less than ϵ.

   Solution: Let i be large enough that $(\frac{1}{2})^i < \epsilon$. Then it is sufficient to repeat the given algorithm i times and output *yes* if and only if the answer was always yes. If there is a perfect matching, the answer is still *yes* all the time and otherwise, the probability that the given algorithm is wrong i times is less than $(\frac{1}{2})^i < \epsilon$.

2. Let A, B and C be $n \times n$ matrices such that $AB \neq C$. Then, for a vector r chosen uniformly at random from $\{0,1\}^n$, show that $\Pr[ABr = Cr] \leq \frac{1}{2}$.

   Hint: Construct a multivariate polynomial p such that $p \equiv 0$ if and only if $AB = C$ and apply the Schwartz-Zippel Lemma.

   Solution: Define the following polynomial of degree 1

   $$p(x) = \|(ABx - Cx)\|_1$$

   for each $x \in R^n$. Then clearly, if $AB = C$ we have $ABx = Cx$ for all x and thus $ABx - Cx = 0$ for all x which implies $p(x) = 0$ for all x. Conversely, if $p(x) = \|ABx - Cx\|_1 = 0$ for all x we get $ABx - Cx = (AB - C)x = 0$ for all x by the fact that the 1-norm is 0 if and only if the vector is 0. In particular, $(AB - C)e_i = 0$ for $i = 1, \ldots, n$. But this is just the i-th column of $(AB - C)$ and we can conclude that $AB = C$.
   By applying the Schwartz-Zippel Lemma with the set $S = \{0,1\}$ we get for r chosen uniformly at random in $\{0,1\}^n$:

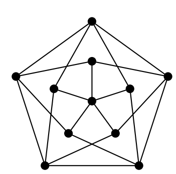
   $$Pr[ABr = Cr] = Pr[p(r) = 0] \leq \frac{1}{|S|} = \frac{1}{2}$$

3. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be any two events. Proof that

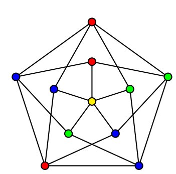
   $$\Pr[\mathcal{E}_1] \leq \Pr[\mathcal{E}_1 | \text{not } \mathcal{E}_2] + \Pr[\mathcal{E}_2]$$

   Solution: By basic probability theory, we get $\Pr[\mathcal{E}_1] = \Pr[\mathcal{E}_1 | \text{not } \mathcal{E}_2] \Pr[\text{not } \mathcal{E}_2] + \Pr[\mathcal{E}_1 | \mathcal{E}_2] \Pr[\mathcal{E}_2]$ and using that all probabilities are between 0 and 1 we get inequality wanted.

4. Determine the chromatic number of the following graph.

   Solution:

The chromatic number is 4 and the picture shows a 4-coloring.
To show that the coloring is optimal, we try to 3-color it. Start with the outer C_5: up to isomorphism there is only one coloring, red-blue-red-blue-green. This forces a red, blue, and green vertex on the inner ring of 5 vertices, which forces a fourth color on the middle vertex.

5. For a graph G, we define $G[X]$, the subgraph *induced* by the vertex set $X \subseteq V(G)$ as the graph with vertex set X that contains all the edges of G with both ends in X.
Prove that $\chi(G) \leq \chi(G[X]) + \chi(G[V \setminus X])$.

Solution: Define $\chi_1 = \chi(G[X])$, $\chi_2 = \chi(G[V \setminus X])$. We prove that there is a valid coloring of G with $\chi_1 + \chi_2$ colors: Color the vertices of X with χ_1 colors such that we have a valid coloring of $G[X]$, and color $V \setminus X$ with χ_2 colors different from the first χ_1 colors so that we get a valid coloring of $G[V \setminus X]$. Note that the edge e of G is either fully included in one of $G[X]$ or $G[V \setminus X]$, or it connects them. In both cases the end vertices of e get different colors: since in the former case both of the induced subgraphs have a valid coloring, and in the latter one, it follows from the fact that the vertices in X get different colors from the vertices in $V \setminus X$.

6. Let G be a graph such that $\chi(G - x - y) = \chi(G) - 2$ for all pairs of distinct vertices $x, y \in V(G)$. Prove that G is the complete graph.

Solution: Suppose by contradiction that G is not complete. Let $x, y \in V(G)$ distinct vertices such that $xy \notin E(G)$. Consider a minimal vertex coloring of $G - x - y$ which uses $\chi(G - x - y)$ colors. Now in G, keep the minimal coloring of $G - x - y$ for all vertices but x and y. Color x and y both with a $(\chi(G - x - y) + 1)$-th color. This coloring is valid for G and only uses $\chi(G - x - y) + 1$ colors, which is a contradiction to the assumption that $\chi(G - x - y) = \chi(G) - 2$.

7. Let G be a graph on n vertices and $\overline{G}$ be its complement. Prove that

   (a) $\chi(G)\chi(\overline{G}) \geq n$.

   (b) $\chi(G) + \chi(\overline{G}) \leq n + 1$.

Solution:

(a) Note that the union $G \cup \overline{G}$ is the complete graph K_n. We construct a valid coloring of K_n with $\chi(G)\chi(\overline{G})$ colors. Then we get the required inequality, since $\chi(K_n) = n$. Denote $V = V(G)$. Let $c : V \to \{1, 2, \ldots, \chi(G)\}$ be a valid coloring of G and $\overline{c} : V \to \{1, 2 \ldots, \chi(\overline{G})\}$ be a valid coloring of $\overline{G}$. Define the coloring $c' : V \to \{1, 2, \ldots, \chi(G)\} \times \{1, 2 \ldots, \chi(\overline{G})\}$ with $c'(v) = (c(v), \overline{c}(v))$ for $v \in V$. It is easy to see that c' is a valid coloring for K_n on V: for distinct vertices $u, v \in V$, if $uv \in E(G)$, then $c(u) \neq c(v)$, and if $uv \in E(\overline{G})$, then $\overline{c}(u) \neq \overline{c}(v)$, both of which imply that $c'(u) \neq c'(v)$.

(b) We prove it by induction on the number of the vertices. It is easy to check the induction basis. Now suppose the inequality holds for all graphs with n vertices, we prove it for the graph G on $n + 1$ vertices. Fix the vertex $v \in V(G)$ and let k be its degree in G, so the degree of v in $\overline{G}$ is $n - k$. Consider the graph $G - v$. Note that adding back v to $G - v$ does not increase the chromatic number if $\chi(G - v) > k$, since one can color v by an existing color different from the colors of its k neighbors; otherwise, it will increase the chromatic number by at most one. The same statement holds for $\overline{G} - v$ with the condition $\chi(\overline{G} - v) > n - k$. Therefore, if at least one of $\chi(G - v) > k$ and $\chi(\overline{G} - v) > n - k$ holds, then applying induction hypothesis to $G - v$ will complete the proof:

$$\chi(G) + \chi(\overline{G}) \leq \chi(G - v) + \chi(\overline{G} - v) + 1 \leq n + 2.$$

Otherwise, we have $\chi(G - v) \leq k$ and $\chi(\overline{G} - v) \leq n - k$, which implies

$$\chi(G) + \chi(\overline{G}) \leq \chi(G - v) + \chi(\overline{G} - v) + 2 \leq k + n - k + 2 = n + 2.$$

This finishes the proof.

8. (a) Show that if an n-vertex graph is d-degenerate, then it has at most dn edges.

   (b) Prove that if the longest path in G has length ℓ, then $\chi(G) \leq \ell + 1$.

Solution:

(a) This can be proved by induction on the number of vertices n. Note that by the induction hypothesis, removing a vertex of degree at most d would result in a graph with at most $d(n-1)$ edges, so the original graph has at most $d(n-1) + d = dn$ edges.

(b) This follows from the fact that any such graph is ℓ-degenerate. To see this, let G' be a subgraph of G and v be an endpoint of a longest path in G'. Since this path cannot be extended, all the neighbors of v in G' are contained in this path, therefore $\deg_{G'}(v) \leq \ell$.