Graph theory - problem set 9

November 14, 2019

1. Deduce the undirected version of Menger's theorem from the directed version.

   Solution: Let G be an undirected graph containing vertices s and t. The "easy" direction of Menger's theorem can be proved with the same argument we have seen in the lecture, so we only need to show the "difficult" direction: if there is no s-t edge (or vertex) separator of size less than k, then there are k edge (or internally vertex) disjoint s-t paths in G.

   So let D be the directed graph obtained from G by replacing every (undirected) edge with two opposite directed edges. There is a bijective correspondence between directed paths in D and undirected paths in G. In particular, if D contained an s-t edge (or vertex) separator of size less than k (whose deletion destroys all directed s-t paths), then deleting the corresponding edges (or vertices) from G would destroy all undirected s-t paths in G. I.e., they would be a separator in G of size less than k, contradicting our assumption. So D has no such separator, either, and we can thus apply Menger's theorem to find the k disjoint directed paths in D. The corresponding paths in G are the ones we were looking for.

2. Let G be a k-connected graph. Show using the definitions that if G' is obtained from G by adding a new vertex V adjacent to at least k vertices of G, then G' is k-connected.

   Solution: Let S be such that $G' - S$ is disconnected. Let us show that $|S| \geqslant k$. Assume the contrary that $|S| \leqslant k - 1$. If $V \in S$, then $G - (S \setminus V)$ is disconnected as well. Since G is k-connected then $|S| > |S \setminus V| \geqslant k$. This is a contradiction. If $V \notin S$ then $G - S$ is connected (by k-connectivity of G) and, since the degree of V is at least k, then V is adjacent for at least one vertex of $G - X$. Hence, $G' - S$ is connected. This is a contradiction.

3. Prove that a graph G on at least $k + 1$ vertices is k-connected if and only if $G - X$ is connected for every vertex set X of size $k - 1$.

   Solution: $\Rightarrow$: By the definition of k-connectivity, if G is k-connected then $G - X$ is connected for every set X of size $k - 1$.

   $\Leftarrow$: Assume the contrary that $G = (V, E)$ is not k-connected. Then there is a set of vertices Y such that $|Y| \leqslant k-1$ and the graph $G-Y$ is disconnected. Hence, there are two vertices x and y, which lie in different connected components. We obtain set Y' from Y by adding $k-1-|Y|$ vertices to Y from $V \setminus \{x, y\}$. Then $G - Y' \supset \{x, y\}$ is a disconnected graph and $|Y'| = k - 1$. This is a contradiction.

4. Show that if G is a graph with $|V(G)| = n \geq k + 1$ and $\delta(G) \geq (n + k - 2)/2$ then G is k-connected.

   Solution: We prove that any two non-adjacent vertices $u, v \in V(G)$ have at least k common neighbor vertices. Then one can easily see that after removing any $k - 1$ vertices from G, if u and v are adjacent, we are done, otherwise they still have at least one common neighbor, so the graph remains connected. Denote the set of neighbor vertices of u, v by $N(u)$, $N(v)$, respectively. Since we have $|N(u) \cup N(v)| \leq n - 2$, we get

   $$n-2 \geq |N(u)| + |N(v)| - |N(u) \cap N(v)| \geq 2 \cdot \frac{n+k-2}{2} - |N(u) \cap N(v)| = n+k-2 - |N(u) \cap N(v)|.$$

   Therefore, we have $k \leq |N(u) \cap N(v)|$.

5. Prove the following variants of Menger's theorem. Let G be a graph and let S, T be disjoint vertex sets. An S-T *path* is a path with one endpoint in S and the other in T. Then:

   (a) The maximum number of edge-disjoint S-T paths equals the min size of an S-T edge separator.

   (b) *[fan lemma]:* If G is k-connected, then for every s and every T of size at least k, there are k vertex-disjoint s-T paths (except at s).

(c) If $|S|, |T| \geq k$ and there is no S-T separator of size $k-1$, then G contains k vertex disjoint S-T paths.
(An S-T separator $X \subseteq V(G)$ is a set such that $G - X$ has no path between $S \setminus X$ and $T \setminus X$.)

Solution:

(a) We construct the graph G' out of G by merging all the vertices in S to a single vertices s and all the ones in T to a single vertex t in such a way that for each vertex $u \in S$, we draw an edge between s and all the neighbors of u in G, allowing multiple edge, and we do the same thing for each $u \in T$ and t. The rest of proof follows by applying Menger's theorem for $s - t$ paths in G'. But note that G' might be a multigraph, if for example two vertices in S share a common neighbor. This version of Menger's theorem still holds for the multigraphs, since one can merge a collection of multiple edges into one edge and then let the capacity of this edge to be the number of multiple edges it represents, and then apply Ford-Fulkerson theorem in the same way as seen in the lecture notes.

(b) We construct the graph G' out of G by adding an extra vertex t to G, and connect t to all vertices in T. By exercise 2, we know that G' is k-connected. By the Global version of Menger's theorem, G' contains k internally vertex-disjoint paths between s and t. Hence, by construction, there are k vertex-disjoint s-T paths (except at s).

(c) The idea is again to construct a graph G' out of G and then apply Menger's theorem to G'. To construct G', we add two extra vertices s, t to G, and connect s to all the vertices in S, and connect t to all the ones in T.

6. Let G be a connected graph with all degrees even. Show that G is 2-edge-connected.

   Solution: As G is connected with all degrees even, it has an Euler tour. Deleting any edge from an Euler tour results in an Euler trail. So $G - e$ has an Euler trail and all its vertices have positive degree, so it is connected. As this is true for any edge e, G is a 2-edge-connected graph.

7. Prove that G is 2-connected if and only if for any three vertices x, y, z there is a path in G from x to z containing y.

   Solution: $\Rightarrow$: We want to show that given x, y, z in G, there exists a path from x to z containing y. The idea is to construct a graph G' out of G and then apply Menger's theorem to G'. To construct G', we add an extra vertex s to G, and connect s to the vertices x and z.
   By exercise 2, G' is 2-connected. By Menger's theorem, there are two internally vertex-disjoint s-y paths in G'. By construction, one of them contains x and another contains z. Therefore, there is a path in G from x to z containing y.

   $\Leftarrow$: Let x be any vertex of G. We want to show that $G - x$ is still connected by showing that any two vertices in $G - x$ are connected. Let y, z be any two vertices of $G - x$. By assumption, there is a path $x \dots y \dots z$ in G. Then there is a path $y \dots z$ in $G - x$, so these two vertices are connected in $G - x$.