1. Deduce the undirected version of Menger’s theorem from the directed version.

**Solution:** Let $G$ be an undirected graph containing vertices $s$ and $t$. The “easy” direction of Menger’s theorem can be proved with the same argument we have seen in the lecture, so we only need to show the “difficult” direction: if there is no $s$-$t$ edge (or vertex) separator of size less than $k$, then there are $k$ edge (or internally vertex) disjoint $s$-$t$ paths in $G$.

So let $D$ be the directed graph obtained from $G$ by replacing every (undirected) edge with two opposite directed edges. There is a bijective correspondence between directed paths in $D$ and undirected paths in $G$. In particular, if $D$ contained an $s$-$t$ edge (or vertex) separator of size less than $k$, then deleting the corresponding directed edges in $D$ would destroy all undirected $s$-$t$ paths in $G$. I.e., they would be a separator in $G$ of size less than $k$, contradicting our assumption. So $D$ has no such separator, either, and we can thus apply Menger’s theorem to find the $k$ disjoint directed paths in $D$. The corresponding paths in $G$ are the ones we were looking for.

2. Let $G$ be a $k$-connected graph. Show using the definitions that if $G'$ is obtained from $G$ by adding a new vertex $V$ adjacent to at least $k$ vertices of $G$, then $G'$ is $k$-connected.

**Solution:** Let $S$ be such that $G' - S$ is disconnected. Let us show that $|S| \geq k$. Assume the contrary that $|S| \leq k - 1$. If $V \in S$, then $G - (S \setminus V)$ is disconnected as well. Since $G$ is $k$-connected then $|S| > |S \setminus V| \geq k$. This is a contradiction. If $V \notin S$ then $G - S$ is connected (by $k$-connectivity of $G$) and, since the degree of $V$ is at least $k$, then $V$ is adjacent for at least one vertex of $G - X$. Hence, $G' - S$ is connected. This is a contradiction.

3. Prove that a graph $G$ on at least $k + 1$ vertices is $k$-connected if and only if $G - X$ is connected for every vertex set $X$ of size $k - 1$.

**Solution:** $\Rightarrow$: By the definition of $k$-connectivity, if $G$ is $k$-connected then $G - X$ is connected for every set $X$ of size $k - 1$.

$\Leftarrow$: Assume the contrary that $G = (V,E)$ is not $k$-connected. Then there is a set of vertices $Y$ such that $|Y| \leq k - 1$ and the graph $G - Y$ is disconnected. Hence, there are two vertices $x$ and $y$, which lie in different connected components. We obtain set $Y'$ from $Y$ by adding $k - 1 - |Y|$ vertices to $Y$ from $V \setminus \{x,y\}$. Then $G - Y' \supset \{x,y\}$ is a disconnected graph and $|Y'| = k - 1$. This is a contradiction.

4. Show that if $G$ is a graph with $|V(G)| = n \geq k + 1$ and $\delta(G) \geq (n + k - 2)/2$ then $G$ is $k$-connected.

**Solution:** We prove that any two non-adjacent vertices $u, v \in V(G)$ have at least $k$ common neighbor vertices. Then one can easily see that after removing any $k - 1$ vertices from $G$, if $u$ and $v$ are adjacent, we are done, otherwise they still have at least one common neighbor, so the graph remains connected. Denote the set of neighbor vertices of $u, v$ by $N(u), N(v)$, respectively. Since we have $|N(u) \cup N(v)| \leq n - 2$, we get

$$n - 2 \geq |N(u)| + |N(v)| - |N(u) \cap N(v)| \geq 2 \cdot \frac{n + k - 2}{2} - |N(u) \cap N(v)| = n + k - 2 - |N(u) \cap N(v)|.$$

Therefore, we have $k \leq |N(u) \cap N(v)|$.

5. Prove the following variants of Menger’s theorem. Let $G$ be a graph and let $S,T$ be disjoint vertex sets. An $S$-$T$ path is a path with one endpoint in $S$ and the other in $T$. Then:

(a) The maximum number of edge-disjoint $S$-$T$ paths equals the min size of an $S$-$T$ edge separator.

(b) [fan lemma]: If $G$ is $k$-connected, then for every $s$ and every $T$ of size at least $k$, there are $k$ vertex-disjoint $s$-$T$ paths (except at $s$).
(c) If |S|, |T| ≥ k and there is no S-T separator of size k − 1, then G contains k vertex disjoint S-T paths.
(An S-T separator X ⊆ V(G) is a set such that G − X has no path between S \ X and T \ X.)

**Solution:**

(a) We construct the graph G′ out of G by merging all the vertices in S to a single vertices s and all the ones in T to a single vertex t in such a way that for each vertex u ∈ S, we draw an edge between s and all the neighbors of u in G, allowing multiple edge, and we do the same thing for each u ∈ T and t. The rest of proof follows by applying Menger’s theorem for s − t paths in G′. But note that G′ might be a multigraph, if for example two vertices in S share a common neighbor. This version of Menger’s theorem still holds for the multigraphs, since one can merge a collection of multiple edges into one edge and then let the capacity of this edge to be the number of multiple edges it represents, and then apply Ford-Fulkerson theorem in the same way as seen in the lecture notes.

(b) We construct the graph G′ out of G by adding an extra vertex t to G, and connect t to all vertices in T. By exercise 2, we know that G′ is k-connected. By the Global version of Menger’s theorem, G′ contains k internally vertex-disjoint s-T paths between s and t. Hence, by construction, there are k vertex-disjoint s-T paths (except at s).

(c) The idea is again to construct a graph G′ out of G and then apply Menger’s theorem to G′. To construct G′, we add two extra vertices s, t to G, and connect s to all the vertices in S, and connect t to all the ones in T.

6. Let G be a connected graph with all degrees even. Show that G is 2-edge-connected.

**Solution:** As G is connected with all degrees even, it has an Euler tour. Deleting any edge from an Euler tour results in an Euler trail. So G − e has an Euler trail and all its vertices have positive degree, so it is connected. As this is true for any edge e, G is a 2-edge-connected graph.

7. Prove that G is 2-connected if and only if for any three vertices x, y, z there is a path in G from x to z containing y.

**Solution:** ⇒: We want to show that given x, y, z in G, there exists a path from x to z containing y. The idea is to construct a graph G′ out of G and then apply Menger’s theorem to G′. To construct G′, we add an extra vertex s to G, and connect s to the vertices x and y. By exercise 2, G′ is 2-connected. By Menger’s theorem, there are two internally vertex-disjoint s-y paths in G′. By construction, one of them contains x and another contains z. Therefore, there is a path in G from x to z containing y.

⇐: Let x be any vertex of G. We want to show that G − x is still connected by showing that any two vertices in G − x are connected. Let y, z be any two vertices of G − x. By assumption, there is a path x . . . y . . . z in G. Then there is a path y . . . z in G − x, so these two vertices are connected in G − x.