1. Determine the edge-chromatic number of the graph below.

2. (a) Find the edge-chromatic number of $K_{2n+1}$ (don’t use Vizing’s theorem).
   (b) Find the edge-chromatic number of $K_{2n}$.

3. Let $G$ be a 3-regular graph with $\chi'(G) = 4$. Prove that $G$ does not have a Hamilton cycle.

4. Prove that every bipartite graph satisfies $\chi'(G) = \Delta(G)$.

5. Deduce the undirected version of Menger’s theorem from the directed version.

6. Let $G$ be a $k$-connected graph. Show using the definitions that if $G'$ is obtained from $G$ by adding a new vertex $V$ adjacent to at least $k$ vertices of $G$, then $G'$ is $k$-connected.

7. Prove that a graph $G$ on at least $k + 1$ vertices is $k$-connected if and only if $G - X$ is connected for every vertex set $X$ of size $k - 1$.

8. Prove the following variants of Menger’s theorem. Let $G$ be a graph and let $S,T$ be disjoint vertex sets. An $S$-$T$ path is a path with one endpoint in $S$ and the other in $T$. Then:
   (a) The maximum number of edge-disjoint $S$-$T$ paths equals the min size of an $S$-$T$ edge separator.
   (b) [fan lemma]: If $G$ is $k$-connected, then for every $s$ and every $T$ of size at least $k$, there are $k$ vertex-disjoint $s$-$T$ paths (except at $s$).
   (c) If $|S|,|T| \geq k$ and there is no $S$-$T$ separator of size $k - 1$, then $G$ contains $k$ vertex disjoint $S$-$T$ paths.
   (An $S$-$T$ separator $X \subseteq V(G)$ is a set such that $G - X$ has no path between $S \setminus X$ and $T \setminus X$.)

9. Let $G$ be a connected graph with all degrees even. Show that $G$ is 2-edge-connected.

10. Prove that $G$ is 2-connected if and only if for any three vertices $x,y,z$ there is a path in $G$ from $x$ to $z$ containing $y$. 