Graph theory - problem set 8

November 7, 2019

Exercises

1. You are given an algorithm testing if a graph G has a perfect matching or not. If G has a perfect matching, it always outputs *yes*. Otherwise, it outputs *yes* only with probability less than $\frac{1}{2}$.
   Given $\epsilon > 0$, use this algorithm to design an algorithm which always outputs *yes* if the graph has a perfect matching and if not, it outputs *yes* only with probability less than ϵ.

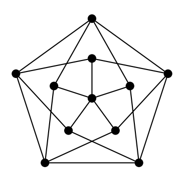
2. Let A, B and C be $n \times n$ matrices such that $AB \neq C$. Then, for a vector r chosen uniformly at random from $\{0,1\}^n$, show that $\Pr[ABr = Cr] \leq \frac{1}{2}$.

   Hint: Construct a multivariate polynomial p such that $p \equiv 0$ if and only if $AB = C$ and apply the Schwartz-Zippel Lemma.

3. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be any two events. Proof that

$$\Pr[\mathcal{E}_1] \leq \Pr[\mathcal{E}_1|\text{not } \mathcal{E}_2] + \Pr[\mathcal{E}_2]$$

4. Determine the chromatic number of the following graph.



5. For a graph G, we define $G[X]$, the subgraph *induced* by the vertex set $X \subseteq V(G)$ as the graph with vertex set X that contains all the edges of G with both ends in X. Prove that

$$\chi(G) \leq \chi(G[X]) + \chi(G[V \setminus X])$$

6. Let G be a graph such that $\chi(G - x - y) = \chi(G) - 2$ for all pairs of distinct vertices $x, y \in V(G)$. Prove that G is the complete graph.

7. Let G be a graph on n vertices and $\overline{G}$ be its complement. Prove that
   (a) $\chi(G)\chi(\overline{G}) \geq n$.
   (b) $\chi(G) + \chi(\overline{G}) \leq n + 1$.

8. (a) Show that if an n-vertex graph is d-degenerate, then it has at most dn edges.
   (b) Prove that if the longest path in G has length ℓ, then $\chi(G) \leq \ell + 1$.