Graph theory - problem set 11

November 28, 2019

1. Let $V$ be a set of $n$ vertices. We will construct a random graph $G$ with $m$ edges. For each $e \in V \times V$ we perform a random experiment, the outcome of which will determine if $e$ is and edge of $G$. The experiments are performed independently, and for every one of them the probability of success is $p$.

What is the probability that we obtain some fixed graph $G_0 = (V, E_0)$, where $|E_0| = m$?

2. Let $(\Omega, \mathbb{P})$ be a probability space. Prove that for any collection of events $E_1, \ldots, E_k$, we have

$$\mathbb{P}\left[\bigcup_{i=1}^{k} E_i\right] \leq \sum_{i=1}^{k} \mathbb{P}[E_i],$$

and if $E_1, \ldots, E_k$ are disjoint events, then we have equality here.

3. Let $\sigma$ be an arbitrary permutation of $\{1, \ldots, n\}$, selected uniformly at random from the set of all permutations (that is, each permutation is selected with probability $\frac{1}{n!}$). What is the expectation of the number of fixed points in $\sigma$? (Recall that $i$ is a fixed point if $\sigma(i) = i$.)

4. Take a complete graph $K_n$ where each edge is independently colored red, green or blue with probability 1/3. What is the expected number of red cliques of size $a$ in this graph?

5. Let $G$ be a graph with $m$ edges, and let $X \subseteq V(G)$ be a random set that contains each vertex of $G$ independently with probability $1/2$. What is the expected number of edges in the induced subgraph $G[X]$?

(Here $G[X]$ is the subgraph of $G$ with vertex set $X$, and contains all edges in $G$ with both ends in $X$.)

6. Let $G$ be a graph with $m$ edges, and let $k$ be a positive integer. Prove that the vertices of $G$ can be colored with $k$ colors in such a way that there are at most $m/k$ monochromatic edges (i.e., edges with both endpoints colored the same).

7. Prove that if $G$ has $2n$ vertices and $e$ edges then it contains a bipartite subgraph with at least $e \frac{n}{2n-1}$ edges. [Use a random partition of the vertices into two parts of size $n$]