Graph theory - problem set 11

November 28, 2019

1. Let V be a set of n vertices. We will construct a random graph G with m edges. For each $e \in V \times V$ we perform a random experiment, the outcome of which will determine if e is and edge of G. The experiments are performed independently, and for every one of them the probability of success is p.

   What is the probability that we obtain some fixed graph $G_0 = (V, E_0)$, where $|E_0| = m$?

2. Let $(\Omega, \mathbb{P})$ be a probability space. Prove that for any collection of events $E_1, \ldots, E_k$, we have

   $$\mathbb{P}\left[\bigcup_{i=1}^{k} E_i\right] \leq \sum_{i=1}^{k} \mathbb{P}[E_i],$$

   and if $E_1, \ldots, E_k$ are disjoint events, then we have equality here.

3. Let σ be an arbitrary permutation of $\{1, \ldots, n\}$, selected uniformly at random from the set of all permutations (that is, each permutation is selected with probability $\frac{1}{n!}$). What is the expectation of the number of fixed points in σ? (Recall that i is a fixed point if $\sigma(i) = i$.)

4. Take a complete graph K_n where each edge is independently colored red, green or blue with probability 1/3. What is the expected number of red cliques of size a in this graph?

5. Let G be a graph with m edges, and let $X \subseteq V(G)$ be a random set that contains each vertex of G independently with probability 1/2. What is the expected number of edges in the induced subgraph $G[X]$?
   (Here $G[X]$ is the subgraph of G with vertex set X, and contains all edges in G with both ends in X.)

6. Let G be a graph with m edges, and let k be a positive integer. Prove that the vertices of G can be colored with k colors in such a way that there are at most m/k monochromatic edges (i.e., edges with both endpoints colored the same).

7. Prove that if G has $2n$ vertices and e edges then it contains a bipartite subgraph with at least $e\frac{n}{2n-1}$ edges. [Use a random partition of the vertices into two parts of size n]