Graph theory - problem set 10

November 21, 2019

1. Deduce the undirected version of Menger's theorem from the directed version.

2. Let G be a k-connected graph. Show using the definitions that if G' is obtained from G by adding a new vertex V adjacent to at least k vertices of G, then G' is k-connected.

3. Prove that a graph G on at least $k+1$ vertices is k-connected if and only if $G - X$ is connected for every vertex set X of size $k-1$.

4. Show that if G is a graph with $|V(G)| = n \geq k+1$ and $\delta(G) \geq (n+k-2)/2$ then G is k-connected.

5. Prove the following variants of Menger's theorem. Let G be a graph and let S, T be disjoint vertex sets. An S-T *path* is a path with one endpoint in S and the other in T. Then:

   (a) The maximum number of edge-disjoint S-T paths equals the min size of an S-T edge separator.

   (b) *[fan lemma]:* If G is k-connected, then for every s and every T of size at least k, there are k vertex-disjoint s-T paths (except at s).

   (c) If $|S|, |T| \geq k$ and there is no S-T separator of size $k-1$, then G contains k vertex disjoint S-T paths.
   (An S-T separator $X \subseteq V(G)$ is a set such that $G - X$ has no path between $S \setminus X$ and $T \setminus X$.)

6. Let G be a connected graph with all degrees even. Show that G is 2-edge-connected.

7. Prove that G is 2-connected if and only if for any three vertices x, y, z there is a path in G from x to z containing y.