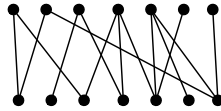


Graph theory - solutions to problem set 6

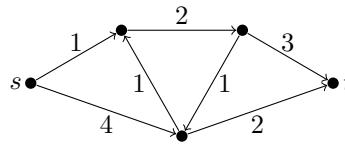
Exercises

1. Find a minimum vertex cover in the following graph.

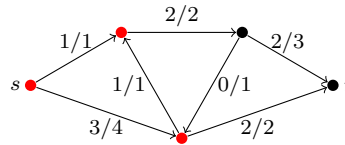


Solution: Since this graph has a perfect matching, the vertices from the top part form a minimum vertex cover.

2. Find a maximum flow from s to t and a minimum s - t cut in the following network.



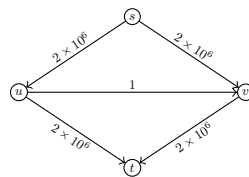
Solution: The only maximum flow here (with value 4) is the following:



An example of a minimum cut (of capacity 4) is shown in red vs black.

3. Construct a network on four vertices for which the Ford-Fulkerson algorithm may need more than a million iterations, depending on the choice of augmenting paths.

Solution: Consider the following network:



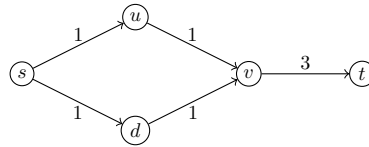
Clearly, the maximum value of a flow is 2 million, but the Ford-Fulkerson algorithm might always choose a path that uses the edge uv by alternately choosing svt and $svut$. This way the value of the flow increases by 1 in every step, so it takes 2 million improvements to reach a maximum flow.

4. Let G be a network with source s , sink t , and integer capacities. Prove or disprove the following statements:

- (a) If all capacities are even then there is a maximal flow f such that $f(e)$ is even for all edges e .
- (b) If all capacities are odd then there is a maximal flow f such that $f(e)$ is odd for all edges e .

Solution:

- (a) Dividing all capacities by two, we obtain a network with integral capacities. Therefore, it has an integral maximum flow. Multiplying this flow by 2, we get an even maximum flow on the original network.
- (b) Counterexample:



5. Let G be a network with source s , sink t , and integer capacities. Prove that an edge e is saturated (i.e., the flow uses its full capacity) in every maximum s - t flow if and only if decreasing the capacity of e by 1 would decrease the maximum value of an s - t flow in G .

Solution: Let k be the value of the maximum flow, i.e., the capacity of a minimum cut in G . We argue by contradiction for both directions.

\Rightarrow : Suppose decreasing the capacity does not decrease the size of the max flow. Then there is a flow in this new network G' of value k that does not use the full capacity of e in G . This corresponds to a flow of size k in G with the same property, contradicting our assumption.

\Leftarrow : Suppose e is not saturated in every maximum s - t flow. If we could assume that this flow is an integer flow, then we would be done: This flow would have value k in G' , contradicting the assumption. But we don't quite know that we can assume this, so let's do something else instead:

Indeed, if e is not saturated in some maximum flow, then e does not occur in any min cut (otherwise the size of this flow would be less than the capacity of a min cut). Now we do know that every cut has an integer capacity, so if e is not in a k -cut, then every cut containing e has capacity at least $k + 1$. Hence decreasing the capacity of e by 1 will not create any cut of size k , therefore the size of the maximum flow does not decrease. Contradiction.

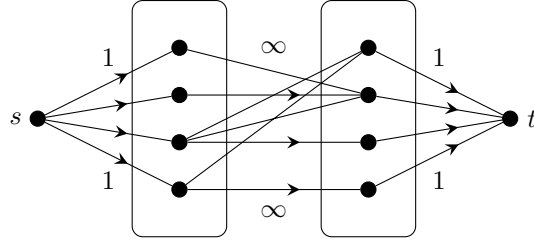
6. Deduce Hall's theorem from the max-flow min-cut theorem.

Solution: Consider the bipartite graph $G = (A \cup B, E)$ that satisfies Hall's conditions. Let us make a network out of G . Add a source s , connect it to all vertices of A by oriented edges of capacity 1. Analogously, add a sink t and connect all vertices of B to it by edges of capacity 1. Let the edges of G be oriented from A to B and their capacities to be infinite. We stress that all the edges are now oriented in the direction from s to t (see the figure below).

If there is a integer flow of value $|A|$ in G , then the edges (x, y) satisfying $x \in A, y \in B$, and $f(x, y) = 1$ constitute a matching of A in G , and we are done. Otherwise, there is a cut (X, Y) of capacity $k < |A|$. We know that

$$|A \cap Y| + |B \cap X| = k < |A| = |A \cap X| + |A \cap Y|,$$

from which we conclude that $|B \cap X| < |A \cap X|$. Let $W = A \cap X$. The set $N(W)$ is contained in $B \cap X$, as otherwise there would be an infinite-capacity edge crossing from X to Y . Thus, $|N(W)| = |B \cap X| < |W|$, and we verified that when a perfect matching does not exist, there is a set W violating Hall's criterion.

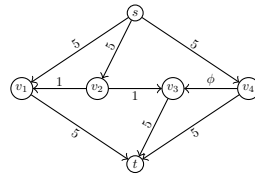


7. Let A be an $n \times m$ matrix of non-negative real numbers such that the sum of the entries is an integer in every row and in every column. Prove that there is an $n \times m$ matrix B of non-negative integers with the same sums as in A , in every row and every column.

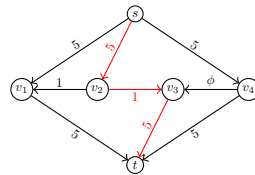
Solution: Let a_1, \dots, a_n and b_1, \dots, b_m be the sums of the entries in the rows and columns of A , respectively. Consider the complete bipartite graph $G = (P \cup Q, E)$, where $P = \{p_1, \dots, p_n\}$ and $Q = \{q_1, \dots, q_m\}$. Let us make a directed network out of G . Orient all edges from P to Q , and set their capacity to infinity. Add a source s , connect it to all vertices of P by oriented edges of capacity a_i for the vertex p_i . Analogously, add a sink t and connect all vertices of Q to it by oriented edges of capacity b_j for the vertex q_j .

Clearly, the minimum capacity of a cut is $a_1 + \dots + a_n = b_1 + \dots + b_m$. Since all capacities are integral, the Ford-Fulkerson theorem shows that there is an integer flow with maximum value. We can then define the entry B_{ij} of the matrix B to be the value of this flow on the edge $p_i q_j$.

8. Let G be a connected graph and let s and t be two vertices in G . Devise a polynomial-time algorithm that finds two *disjoint* sets S and T such that $s \in S$, $t \in T$, and $\delta(S) + \delta(T)$ is minimum, where $\delta(X)$ is the number of edges between X and $V - X$.
9. Prove that the Ford-Fulkerson algorithm might not stop on the following network with the capacities shown on the edges, where $\phi = \frac{\sqrt{5}-1}{2}$. For this, use induction to show that the “residual capacities” $c(u, v) - f(u, v)$ on the three horizontal edges can be $\phi^k, 0, \phi^{k+1}$, for every k . (Note that $\phi^2 = 1 - \phi$.)



Solution: Suppose the Ford-Fulkerson algorithm starts by choosing the central augmenting path, shown in the figure below.



The three horizontal edges, in order from left to right, now have residual capacities 1, 0, and ϕ . Suppose inductively that the horizontal residual capacities are $\phi^{k-1}, 0, \phi^k$ for some positive integer k .

1. Augment along B , adding ϕ^k to the flow; the residual capacities are now $\phi^{k+1}, \phi^k, 0$.
2. Augment along C , adding ϕ^k to the flow; the residual capacities are now $\phi^{k+1}, 0, \phi^k$.
3. Augment along B , adding ϕ^{k+1} to the flow; the residual capacities are now $0, \phi^{k+1}, \phi^{k+2}$.
4. Augment along A , adding ϕ^{k+1} to the flow; the residual capacities are now $\phi^{k+1}, 0, \phi^{k+2}$.

