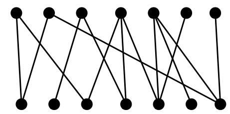
Graph theory - solutions to problem set 5

1. Find a maximum matching in the following graph.



   Solution: It has a perfect matching.

2. Construct a 2-regular graph without a perfect matching.

   Solution: An odd cycle.

3. Construct preference lists for the vertices of $K_{3,3}$ so that there are multiple stable matchings.

   Solution: For example, for parts $\{1, 2, 3\}$ and $\{x, y, z\}$, take

$$\begin{pmatrix} 1: & z & y & x \\ 2: & y & x & z \\ 3: & x & z & y \end{pmatrix} \qquad \begin{pmatrix} x: & 2 & 1 & 3 \\ y: & 3 & 1 & 2 \\ z: & 1 & 2 & 3 \end{pmatrix}$$

   Here $\{(1, z), (2, y), (3, x)\}$ is a stable matching (optimal for the numbers), and another one is $\{(1, z), (2, x), (3, y)\}$ (optimal for the letters).

4. The Gale-Shapley algorithm produces stable matchings on complete bipartite graphs. Consider now complete graphs. Are all matchings stable?

   Solution: No. Consider K_4 with vertices $\{1, 2, 3, 4\}$ and preferences:

$$\begin{pmatrix} 1: & 3 & 2 & 4 \\ 2: & 1 & 3 & 4 \\ 3: & 2 & 1 & 4 \\ 4: & 2 & 1 & 3 \end{pmatrix}$$

   Vertex 4 will have to be paired with someone, let's say i, leaving only one other pair, let's say j, k who are paired together. i prefers j or k over 4; also either j or k prefers i over its actual partner. Hence no matching is stable.

5. Describe an algorithm for finding a maximum cardinality matching in a bipartite graph.

   Solution: See Augmenting Path Algorithm in lecture notes.

6. Show that if $G = (A \cup B, E)$ is a bipartite graph such that $|N(S)| \geq |S| - d$ holds for every $S \subseteq A$, then G has a matching with at least $|A| - d$ edges.

   Solution: Add d new vertices to B, each connected to all vertices in A; let G' be the new graph. Then G' has $|N_{G'}(S)| \geq |S|$ for every $S \subset A$ (S has at least $|S| - d$ neighbors from G, and is connected to the d new vertices). By Hall's Theorem, G' has a matching for A, which has $|A|$ edges. At most d of these edges contain a new vertex of G', which leaves at least $|A| - d$ edges from G.

7. An $r \times s$ *Latin rectangle* is an $r \times s$ matrix A with entries in $\{1, \dots, s\}$ such that each integer occurs at most once in each row and at most once in each column. An $s \times s$ Latin rectangle is called a *Latin square*. Prove that every $r \times s$ Latin rectangle can be extended to an $s \times s$ Latin square.

   Solution: Define a bipartite graph whose vertex set consists of two copies of $\{1, \dots, s\}$, call them S_1 and S_2. We connect $i \in S_1$ with $j \in S_2$ if the i-th column of the $r \times s$ Latin rectangle does not contain the number j. What we are looking for is a matching that matches S_1, since then we can put numbers on row $r + 1$ such that no number is repeated in that row, and no number is repeated in a column.

To see if such a matching exists we use Hall's Theorem, or more specifically Problem 1 above. A column $i \in S_1$ contains r distinct numbers, so there are $s - r$ numbers that it does not contain. That means that the vertex $i \in S_1$ has degree $s - r$. On the other hand, a number $j \in S_2$ occurs exactly once in each of the r rows, and at most once in any of the s columns. Hence there are $s - r$ columns that do not contain j, so the degree of $j \in S_2$ is $s - r$. Therefore, the graph is $(s - r)$-regular, so by Problem 6, there is a perfect matching.

8. Let G be a bipartite graph with parts of size $2n$ and minimum degree at least n. Prove that G has a perfect matching.

   Solution: Let G have parts A and B. We will check Hall's condition for A. Take $X \subseteq A$. If X is empty, then $|N(X)| = |X| = 0$, so the condition is satisfied. If $1 \leq |X| \leq n$, then $|N(X)| \geq n \geq |X|$, because any vertex in X has at least n neighbors in B. Finally, if $|X| > n$, then $N(X) = B$ because every vertex v in B has at least n neighbors in A, so it must have a neighbor in X, as well. (Otherwise $X \cup N(v)$ would contain more than $2n$ vertices in A). In particular, $|N(X)| = 2n \geq |X|$, so the condition holds for every X. By Hall's theorem, there is a perfect matching.

9. Give a graph-theoretic proof of the following statement: if there exist injections $f : A \to B$, $g : B \to A$ between infinite sets A and B, then there exists a bijection $h : A \to B$.

   Solution: We visualize A and B as sets of blue and red vertices: $V = A \cup B$. We then define the following directed edges: $\forall a \in A$ add $(a, f(a))$ as blue edges, and $\forall b \in B$ add $(b, g(b))$ as red edges.

   Then we can partition the resulting graph G into a union of connected components. These connected components can be of three different types:

   - finite directed cycle: $v_1 \to v_2 \to \ldots \to v_n \to v_1$
   - doubly-infinite path, a path that has no beginning nor end: $\ldots \to v_{-1} \to v_0 \to v_1 \to \ldots$
   - singly-infinite path, a path with a beginning but no end: $v_0 \to v_1 \to v_2 \to \ldots$

   In the case of a cycle or a doubly-infinite path, the blue arcs define a one-to-one correspondence between the blue vertices of the component and the red vertices.

   In the case of a singly-infinite path, the blue edges will still determine a one-to-one correspondence between the blue and red vertices of the path if the path begins with a blue vertex, but not if the path begins with a red vertex. However in this latter case we can take the red edges instead. Thus we can pair up the vertices of A and B along each connected component, and the union of these correspondences determines a one-to-one correspondence between A and B.