Graph theory - solutions to problem set 4

1. In this exercise we show that the sufficient conditions for Hamiltonicity that we saw in the lecture are "tight" in some sense.

   (a) For every $n \geq 2$, find a non-Hamiltonian graph on n vertices that has $\binom{n-1}{2} + 1$ edges.

   Solution: Consider the complete graph on $n-1$ vertices K_{n-1}. Add a new vertex v and connect it to a vertex $V(K_{n-1})$. This graph has $\binom{n-1}{2} + 1$ edges and it is non-Hamiltonian: every cycle uses 2 edges at each vertex, but v has only one adjacent edge.

   (b) For every $n \geq 2$, find a non-Hamiltonian graph on n vertices that has minimum degree $\lceil \frac{n}{2} \rceil - 1$.

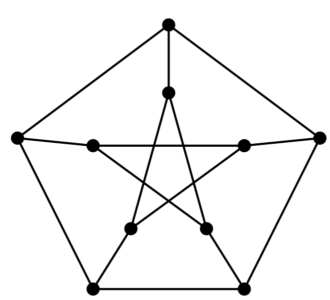
   Solution: Let G_1 be a complete graph on $\lceil \frac{n}{2} \rceil$ vertices and G_2 be a complete graph on $\lfloor \frac{n}{2} \rfloor$ vertices which is disjoint from G_1. Fix a vertex $v \in V(G_1)$ and connect it to all the vertices of G_2. Let G be the resulting graph: it has minimum degree $\lceil \frac{n}{2} \rceil - 1$ and it is non-Hamiltonian, since every cycle passing through all the vertices of G has to pass through v at least twice.

   (c) For every $k, n \geq 2$, find a graph G on *at least* n vertices such that $\delta(G) = k$ but G contains no cycle longer than $k+1$. **Solution:** Let $a = \lceil n/k \rceil$, and take a disjoint copies of K_{k+1}. Then this graph has $a(k+1) > n$ vertices, each of them has degree k, but there is no cycle longer than $k+1$. We can actually find a connected such graph if join all of these cliques at one vertex, creating a star of cliques.

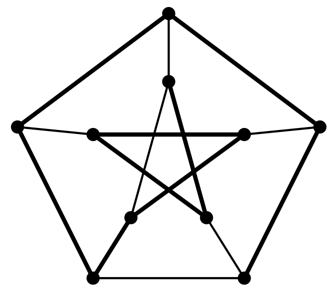
2. Check that the proof of Dirac's Theorem also proves the following statement (called Ore's theorem): If for all non-adjacent vertices u, v in an n-vertex graph G we have $d(u) + d(v) \geq n$, then G has a Hamilton cycle.

   Solution: There were two places in the proof of Dirac's Theorem where we used the condition that $\delta(G) \geq \frac{n}{2}$: To show that G is connected, and to show that there is an edge in P that is both type-1 and type-2. The proof of the first statement is very similar in this case: if u and v were in different components, then the component of u would contain at least $d(u)+1$ vertices, and the component of v would have at least $d(v)+1$ vertices, which would give more than n vertices in total. The second statement follows because there are $d(v_1)$ type-1 edges and $d(v_k)$ type-2 edges. But then if v_1 and v_k are not adjacent, then P has $n-1 < d(v_1) + d(v_k)$ edges by assumption, so some edge is both type-1 and type-2 and we can continue the argument. Otherwise, we get a cycle $v_1 \dots v_k v_1$ that we can use as C for the rest of the proof.

3. The graph below is called the Petersen graph. Does it have a Hamilton path? And a Hamilton cycle?



   Solution: Here is a Hamilton path:



Let us show that there is no Hamilton cycle in the Petersen graph P. One can check that the girth of P is 5 (i.e. P has no 3-cycle or 4-cycle).

Assume there is a Hamilton cycle C in P. Since C must go through each vertex, C is actually C_{10} (i.e. the Petersen graph contains C_{10}). Then there are five more edges in P. If each of the latter edges connects two opposite vertices on C, then there is a 4-cycle. Therefore, some edge e joins vertices at distance 4 in C (why cannot it be 2 or 3?). Let e be incident to vertices A and B, and D be the opposite vertex to A in C. The vertex D must be connected to one of the neighbours of A in C (Why?), let us call it F. Then $ABDFA$ is a 4-cycle. This is a contradiction.

4. Show the following two properties of Minimum Spanning Trees (MST), under the assumption that no two edge weights are equal.

   (a) **Cut Property:** the smallest edge crossing any cut must be in all MSTs. Reminder: a cut in a graph $G = (V, E)$ is a partition $A \cup B = V$.

   (b) **Cycle Property:** The largest edge on any cycle is never in any MST.

   Solution:

   (a) Let $G = (V, E)$ be a graph, $A \cup B = V$ a cut of this graph and let T be a spanning tree of G. Let $e = (u, v)$ be the edge of minimum weight that crosses from A to B. Suppose now that T does not contain e. Since T is connected, it contains a path P between u and v, and P must contain an edge f that crosses from A to B. But $T' = T - f + e$ is a tree of smaller weight, which is a contradiction.

   (b) Let C be a cycle in G and let $e = (u, v)$ be the edge of maximum weight on C. Suppose it is contained in an MST T. Deleting e from T we get two connected components T_1, T_2. But the cycle C must contain an edge $f \neq e$ with one end in T_1 and one end in T_2, and $T - e + f$ is a tree of smaller weight. Contradiction.

5. Let $G = (V, E)$ be a graph with weights $w : E \to \mathbb{R}$. Consider the problem of identifying a forest of maximum weight in G. Show that this problem can be reduced to the problem of computing a minimum weight spanning tree in a suitable graph G' with weights w'. Is your reduction efficient in the sense that G' is of polynomial size in G?

   Solution: We construct a new graph $G' = (V', E')$ with edge weight $w' : E' \to \mathbb{R}$ such that $V \subseteq V'$, $E \subseteq E'$ and a minimum weight spanning tree of G' restricted to G is a maximum weight forest of G. Let $V' = V \cup \{u\}$ where u is a new vertex and let $E' = E \cup \{(u, v) : v \in V\}$ be the new edge set. Define $w'(e) = -w(e)$ for all edges $e \in E$ and $w'(e) = 0$ otherwise. Let T be a spanning tree of G' and F its restriction to G. Then if F is not of maximal weight, there exists an other forest F' with higher weight. Adding an edge from u to each connected component of F' gives a tree in G' whose weight is lower than the weight of T. This implies that T was not a minimal weight spanning tree and thus each minimal weight spanning tree of G' gives a maximal weight forest in G.

6. (a) Show that a k-regular graph with girth 5 must have at least $k^2 + 1$ vertices.

   (b) Find a k-regular graph with girth 5 and $k^2 + 1$ vertices for $k = 2, 3$.

   (c) Show that a k-regular graph with girth 4 must have at least $2k$ vertices.

   Solution:

(a) Take one vertex v and its k neighbors N. These neighbors are not connected, since otherwise there would be a triangle. Further, non of the vertices in N share a neighbor other than v, since otherwise there would be a four-cycle. This implies that each of the k vertices in N has $k-1$ new neighbors (they are of distance two to v).
In total, we get $1+k+k(k-1) = 1+k+k^2-k = k^2+1$ vertices.

(b) For $k=2$ the 5-cycle is an example and for $k=3$ the Petersen graph.

(c) Take two adjacent vertices. They can not have a common neighbor, since otherwise there would be a triangle. So each of them has $k-1$ new neighbors.
Together, this are $2+2(k-1) = 2k$ vertices.

7. Use Ore's theorem to give a short proof of the fact that any n-vertex graph G with more than $\binom{n-1}{2}+1$ has a Hamilton cycle.

Solution: Set $|V(G)| = n$. If the graph is complete then it has a Hamilton cycle. Consider two non-adjacent vertices u, v. Remove them to get a graph $G-u-v$ with $n-2$ vertices and $|E(G)|-d(u)-d(v)$ edges. It has at most $\binom{n-2}{2}$ edges, so we have

$$\binom{n-2}{2} \geq |E(G)| - d(u) - d(v) > \binom{n-1}{2} + 1 - d(u) - d(v)$$

$$\Longrightarrow \quad d(u) + d(v) > \binom{n-1}{2} - \binom{n-2}{2} + 1 = n-1.$$

Thus the condition of Exercise 2 holds, so G has a Hamilton cycle.

8. Let G be a connected graph on n vertices with minimum degree δ. Show that

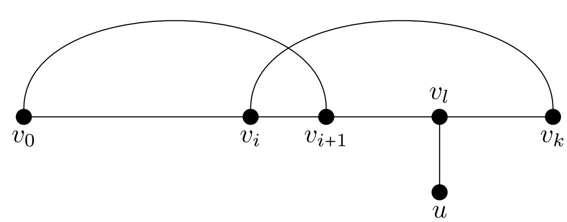
(a) if $\delta \leq \frac{n-1}{2}$ then G contains a path of length 2δ, and

(b) if $\delta \geq \frac{n-1}{2}$ then G contains a Hamiltonian path.

Solution: We show that G contains a path of length at least $\min\{2\delta, n-1\}$.Take a longest path $P = v_0 v_1 \cdots v_k$. If $k = n-1$, then we are done, so we can assume that there is a vertex not on P. Then by connectedness there must be a vertex u that is not on P but adjacent to a vertex of P, let's say v_l.

Observe that v_l is not v_0 or v_k, since then we could extend P to a longer path. Also, v_0 and v_k are not adjacent, otherwise $u v_l \cdots v_k v_0 \cdots v_{l-1}$ would be a longer path. And the neighbourhoods $N(v_0)$ and $N(v_k)$ are contained in $\{v_1, \ldots, v_k\}$, again since otherwise we could the extend P.

One more observation, which is a bit trickier: we cannot have anything like (not showing all the vertices on P)



since this would also give a longer path:

$$v_{l+1} \cdots v_k v_i \cdots v_0 v_{i+1} \cdots v_l u.$$

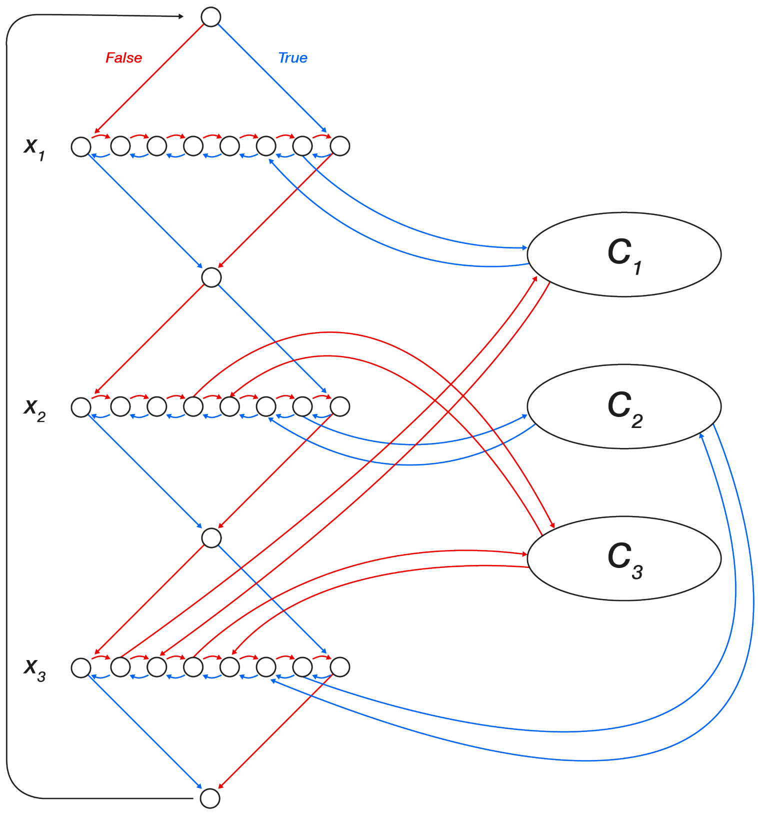
Note that something similar works when $i > l$. Thus we cannot have v_i adjacent to v_k and at the same time v_{i+1} adjacent to v_0.

The set $\{v_1, \ldots, v_k\}$ contains the two sets $N(v_0)$ and $\{v_{i+1} : v_i \in N(v_k)\}$, both of which have size at least δ. Our last observation implies that these two sets are disjoint, which tells us that $k \geq 2\delta$.

9. Construct a directed graph that has a directed Hamilton cycle if and only if the following formula has a satisfying assignment:

$$\{x_1, \bar{x}_3, x_4\} \wedge \{x_2, x_3, \bar{x}_4\} \wedge \{\bar{x}_2, \bar{x}_3\}.$$

Solution: To keep the drawing of the construction simple, we solve the exercise for $C_1 \wedge C_2 \wedge C_3$ where $C_1 := \{x_1, \bar{x}_3\}, C_2 := \{x_2, x_3\}, C_3 := \{\bar{x}_2, \bar{x}_3\}$. The following figure shows the construction. Add a fourth module to extend the construction to the variable x_4.



10. Prove that if a tournament contains a directed cycle (i.e., it is not the transitive tournament) then it contains a directed triangle (3-cycle), as well.

Solution: Take a shortest directed cycle in the tournament $C = v_1 \dots v_k$. If $k > 3$ then C has a "diagonal": v_1 and v_3 are connected by an edge in some direction. If $v_1 \to v_3$ then $v_1 v_3 v_4 \dots v_k$ is a directed cycle of length $k-1$. If $v_3 \to v_1$ then $v_1 v_2 v_3$ is a directed cycle of length 3. Either way, there is a shorter cycle, contradicting our assumption.

11. We say that a vertex u in a tournament is *almost central* if for every other vertex v, there is a directed u-v path of length at most 2. Prove that every tournament has an almost central vertex.

Solution: We proceed by induction on the number of vertices. For $n = 2$, it is clear.
Let $n > 2$ and let $G = (V, A)$ be a tournament on n vertices. Pick a vertex w and consider the tournament $G' = G - w$ on the set V' of $n - 1$ vertices. By induction, there exists an almost central vertex v in G'. The situation now breaks down in three cases.
If there is an arc $a \in A$ directed from v to w, then v is almost central in G. Otherwise, consider the sets $V_i' := \{u \in V' \mid d(v, u) = i\}, i = 1, 2,$ of vertices at distance i of v. Note that because v is almost central in G', we have that $\{v\} \cup V_1' \cup V_2' = V'$.
If there is an arc directed from V_1' to w, the v is almost central in G. Otherwise, all arcs between w and V_1' are directed from w to V_1' and there is a directed path of length at most 2 from w to all vertices in $V_1' \cup V_2'$. Furthermore, w and v are connected by an arc from w to v. Thus w is almost central in G.