Graph theory - solutions to problem set 2

Exercises

1. Prove the triangle-inequality in graphs: for any three vertices u, v, w in a graph G,

$$d(u, v) + d(v, w) \geq d(u, w).$$

   Solution. If $d(u, v) = \infty$ or $d(v, w) = \infty$, there is nothing to prove.

   Otherwise, according to the definition of the distance, there is a u-v path of the length $d(u, v)$ and a v-w path of the length $d(v, w)$. Joining them together we obtain the u-w walk of the length $d(u, v)+d(v, w)$. We have seen in class, that this walk will then contain a u-w path, which is clearly not longer than the walk. Therefore, the shortest u-w path is no longer than $d(u, v) + d(v, w)$.

2. Show that a graph is connected if and only if it contains a spanning tree.

   Solution. If there is a spanning tree then the graph is clearly connected: for any vertices u and v, there will be a u-v path in the tree, hence in the graph, as well. If the graph is connected then the BFS algorithm finds a spanning tree, and this proves that a spanning tree exists.

3. Prove that a forest on n vertices with c connected components has exactly $n - c$ edges.

   Solution. Let $T_1, \ldots, T_c$ be the components of the forest, on $n_1, \ldots, n_c$ vertices, respectively. Each T_i itself is a connected acyclic graph, hence it is a tree (considered as a graph on its own). Therefore, T_i contains $n_i - 1$ edges for each i. Altogether, the graph contains $\sum_{i=1}^{c}(n_i - 1) = \sum_{i=1}^{c} n_i - c = n - c$ edges.

4. Let T be a tree and e be an edge of T. Prove that $T - e$ is not connected.

   Solution. Let $e = uv$ and suppose $T - e$ is connected. Then, in particular, $T - e$ contains a u-v path P. But then $P + e$ is a cycle in T, a contradiction.

5. Let T be a tree and let u and v be two non-adjacent vertices of T. Prove that $T + uv$ contains a unique cycle.

   Solution. Since T is a tree, it is a connected graph without cycles. Then, T contains a path between u and v and adding the edge uv gives a cycle in $T + uv$.
   For uniqueness, suppose by contradiction that $T + uv$ contains at least two cycles C_1 and C_2. Since T does not have a cycle, we may suppose that uv is contained in both cycles. By removing uv we get a closed walk, which contains a cycle.

6. Let W be a closed walk that uses the edge e exactly once. Prove that W contains a cycle through e.

   Solution. Let $v_1 v_2 \ldots v_n v_1$ be a shortest closed walk that uses the edge e exactly once. We claim that this walk is a cycle. Indeed, if $v_i = v_j$ for some $i < j$, then either the closed walk $v_1 \ldots v_i v_{j+1} \ldots v_1$ or the closed walk $v_i v_{i+1} \ldots v_j$ uses the edge e exactly once, and both of them are shorter, which is not possible. (Why doesn't this argument work for an arbitrary walk that uses the edge e exactly twice?)

7. Prove that every connected graph on $n \geq 2$ vertices has a vertex that can be removed without disconnecting the remaining graph.

   Solution. Take a spanning tree T of the graph. It has at least two leaves, say x and y. Then $T - x$ and $T - y$ are both connected, hence so are their supergraphs, $G - x$ and $G - y$.

8. Show that every tree T has at least $\Delta(T)$ leaves. (Where $\Delta(T)$ is the maximum degree of T.)

   Solution. Let v be a vertex with degree $d = \Delta(T)$. For every edge vw incident to v, take a longest path starting with vw. By maximality (as in the proof that every tree has a leaf), the last vertex of this path is a leaf. Doing this for each of the d edges incident to v, we get d paths starting at v, which are disjoint except for v (otherwise we would get a cycle). Thus each path gives a different leaf, and we get $d = \Delta(T)$ leaves.

Alternative solution. If you remove v and its incident edges, you are left with d connected components $T_1, \dots, T_d$, each of which is a tree. By a lemma from class, every tree with at least two vertices has at least two leaves. Hence the T_i with at least two vertices have at least two leaves, one of which must be a leaf of T (one of the two leaves might have been adjacent to v, but not both because that would give a cycle). Some of the T_i might be single vertices, in which case those vertices were leaves in T (they must have been adjacent to v and to no other vertex).

9. Let T be a tree on t vertices and suppose G is a graph with $\delta(G) \geq t - 1$. Show that $T \subseteq G$, i.e., G has a subgraph isomorphic to T.

   Solution. We proceed by induction on t. For $t = 1$ the tree consists only of a vertex and is thus contained in any graph.
   Suppose that the statement is true for t, and let T be a tree on $t+1$ vertices, G a graph with $\delta(G) \geq t$. Let v be a leaf of T which is connected to u in T. Consider $T - v$, obtained by removing v from T, which is a tree on t vertices. Then by the induction hypothesis, there exists an inclusion $i : (T - v) \to G$. The vertex $i(u)$ has degree at least t and the image $i(T - v)$ has t vertices. This implies that there exists a neighbor w of $i(u)$ which is not contained in $i(T - v)$. Expand i to T by defining $i(v) = w$.

10. Let G be a graph on n vertices. Prove that if G has at least $2n - 1$ edges, then it contains an even cycle.

    Solution. On the lecture we proved that G contains a bipartite subgraph H with $|E(H)| \geqslant \frac{|E(G)|}{2}$. We have that $|V(H)| \leq n$ and $|E(H)| \geq n$ in H. Therefore, H contains a cycle (otherwise H is a forest, and the number of edges in a forest is strictly less than the number of vertices). Since H is a bipartite graph, this cycle has even length.