# Graph theory - problem set 7

October 31, 2019

1. Show that Hall's theorem can be derived from König's theorem.

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3. (*Flow Decomposition*) Show that we can decompose any feasible flow on a network $G = (V, A)$ into at most $|A|$ path and cycles.

4. (*Halloweeeeen*) On the 31st of October, enthusiastic partying at a prominent Swiss university has unfortunately resulted in the arrival at the university's medical clinic of 169 students in need of emergency treatment. Each of the 169 students requires a transfusion of one unit of blood. The clinic has supplies of 170 units of blood. The number of units of blood available in each of the four major blood groups and the distribution of patients among the groups in summarized below:

| *Blood type* | $A$ | $B$ | $O$ | $AB$ |
|---|---|---|---|---|
| *Supply* | 46 | 34 | 45 | 45 |
| *Demand* | 39 | 38 | 42 | 50 |

$A$ patients can only receive $A$ or $O$ blood. $B$ patients can only receive $A$ or $O$ blood. $O$ patients can receive only $O$ blood. $AB$ patients can receive any of the four blood types.

(a) Give a max-flow formulation that determines a distribution that satisfies the demands of a maximum number of patients. You should draw a directed graph with edge capacities such that a feasible flow corresponds to a feasible choice for the transfusion.

(b) Find a cut in your graph of value smaller than 169. Use it to give an explanation of why not all of the patients can receive blood from the available supply. Try to make your explanation understandable to the clinic's staff, who do not know network flow theory.

5. Let $G = (V, A)$ be a directed graph and let us fix an origin node $s \in V$ and a destination node $t \in V$. We define the *connectivity* of a graph as the maximum number of vertex-disjoint (besides $s$ and $t$) directed paths from $s$ to $t$. We define the *vulnerability* of the graph as the minimum number of vertices (besides $s$ and $t$) that need to be removed so that there exists no directed path from $s$ to $t$. Prove that connectivity is equal to vulnerability.

6. The *Edmonds-Karp algorithm* is the same as Ford-Fulkerson, except it always chooses an augmenting path of shortest length (this can be found using BFS). In this problem we show that this small requirement can significantly improve the running time of the algorithm: no matter what the capacities are (large or irrational), it finds a max flow in polynomial time (in $O(|V||E|^2)$ steps, to be precise).

(a) Let $l_a(v)$ denote the length of the shortest augmenting path from $s$ to $v$ after $a$ steps. Show that if the algorithm chooses the augmenting $s$-$t$ path $v_0 \dots v_k$ after step $a$, then $l_a(v_{i+1}) = l_a(v_i) + 1$ for every $0 \leq i \leq k - 1$.

(b) Prove that these distances cannot decrease, i.e., $l_a(v) \geq l_{a-1}(v)$ for every $a$ and $v$.
[Hint: for fixed $a$, choose a $v$ such that $l_a(v) < l_{a-1}(v)$ and $l_a(v)$ is minimum]

(c) Show that if $l_i(t) = l_a(t)$ for some $i \geq a$, then after step $i$ the algorithm saturates an edge $uv$ such that $l_a(v) = l_i(v) = l_i(u) + 1 = l_a(u) + 1$ and does not unsaturate any other edge with this property. We get $l_{a+|E|}(t) > l_a(t)$ by noting that there are at most $|E|$ edges to saturate.
[Hint: Look at the augmenting path. Use (b) on $l_i$ and on the analogously defined $m_i(v)$ for the shortest length of an augmenting $v$-$t$ path (i.e., that $m_a(v) \geq m_{a-1}(v)$)]

(d) Deduce that the algorithm stops after at most $|V||E|$ improvements. We get a $O(|V||E|^2)$ running time using a $O(|E|)$-time BFS to find augmenting paths.

7. Suppose you are given an oracle that, given a graph $G$, tells you wether $G$ has a perfect matching or not. Show how to use this oracle to determine the maximum cardinality matching of a graph $G(V, E)$. The total number of calls should be at most $|V| + |E|$. *Hint: modify the graph at each call of the oracle.*