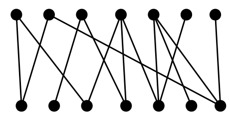
Graph theory - problem set 5

October 16, 2019

1. Find a maximum matching in the following graph.



2. Construct a 2-regular graph without a perfect matching.
3. Construct preference lists for the vertices of $K_{3,3}$ so that there are multiple stable matchings.
4. The Gale-Shapley algorithm produces stable matchings on complete bipartite graphs. Consider now complete graphs on an even number of vertices. Are all matchings stable?
5. Describe an algorithm for finding a maximum cardinality matching in a bipartite graph.
6. Show that if $G = (A \cup B, E)$ is a bipartite graph such that $|N(S)| \geq |S| - d$ holds for every $S \subseteq A$, then G has a matching with at least $|A| - d$ edges.
7. An $r \times s$ *Latin rectangle* is an $r \times s$ matrix A with entries in $\{1, \ldots, s\}$ such that each integer occurs at most once in each row and at most once in each column. An $s \times s$ Latin rectangle is called a *Latin square*. Prove that every $r \times s$ Latin rectangle can be extended to an $s \times s$ Latin square.
8. Let G be a bipartite graph with parts of size $2n$ and minimum degree at least n. Prove that G has a perfect matching.
9. Give a graph-theoretic proof of the following statement: if there exist injections $f : A \to B$, $g : B \to A$ between infinite sets A and B, then there exists a bijection $h : A \to B$.