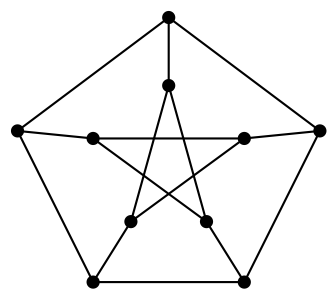
Graph theory - problem set 4

October 10, 2019

Exercises

1. In this exercise we show that the sufficient conditions for Hamiltonicity that we saw in the lecture are 'tight' in some sense.
   (a) For every $n \geq 2$, find a non-Hamiltonian graph on n vertices that has $\binom{n-1}{2} + 1$ edges.
   (b) For every $n \geq 2$, find a non-Hamiltonian graph on n vertices that has minimum degree $\lceil \frac{n}{2} \rceil - 1$.
   (c) For every $k, n \geq 2$, find a graph G on *at least* n vertices such that $\delta(G) = k$ but G contains no cycle longer than $k + 1$.
2. Check that the proof of Dirac's Theorem also proves the following statement (called Ore's theorem): If for all non-adjacent vertices u, v in an n-vertex graph G we have $d(u) + d(v) \geq n$, then G has a Hamilton cycle.
3. The graph below is called the Petersen graph. Does it have a Hamilton path? And a Hamilton cycle?



4. Show the following two properties of Minimum Spanning Trees (MST), under the assumption that no two edge weights are equal.
   (a) **Cut Property:** the smallest edge crossing any cut must be in all MSTs. Reminder: a cut in a graph $G = (V, E)$ is a partition $A \cup B = V$.
   (b) **Cycle Property:** The largest edge on any cycle is never in any MST.
5. Let $G = (V, E)$ be a graph with weights $w : E \to \mathbb{R}$. Consider the problem of identifying a forest of maximum weight in G. Show that this problem can be reduced to the problem of computing a minimum weight spanning tree in a suitable graph G' with weights w'. Is your reduction efficient in the sense that G' is of polynomial size in G?
6. (a) Show that a k-regular graph with girth 5 must have at least $k^2 + 1$ vertices.
   (b) Find a k-regular graph with girth 5 and $k^2 + 1$ vertices for $k = 2, 3$.
   (c) Show that a k-regular graph with girth 4 must have at least $2k$ vertices.
7. Use Ore's theorem from Exercise 2 to give a short proof of the fact that any n-vertex graph G with more than $\binom{n-1}{2} + 1$ has a Hamilton cycle.
8. Let G be a connected graph on n vertices with minimum degree δ. Show that
   (a) if $\delta \leq \frac{n-1}{2}$ then G contains a path of length 2δ, and
   (b) if $\delta \geq \frac{n-1}{2}$ then G contains a Hamiltonian path.

9. Construct a directed graph that has a directed Hamilton cycle if and only if the following formula has a satisfying assignment:

$$\{x_1, \bar{x}_3, x_4\} \wedge \{x_2, x_3, \bar{x}_4\} \wedge \{\bar{x}_2, \bar{x}_3\}.$$

10. Prove that if a tournament contains a directed cycle (i.e., it is not the transitive tournament) then it contains a directed triangle (3-cycle), as well.

11. We say that a vertex u in a tournament is *almost central* if for every other vertex v, there is a directed u-v path of length at most 2. Prove that every tournament has an almost central vertex.