Graph theory - solutions to problem set 3

Exercises

1. For what values of n does the graph K_n contain an Euler trail? An Euler tour? A Hamilton path? A Hamilton cycle?

   Solution:

   Euler trail: K_1, K_2, and K_n for all odd $n \geq 3$.
   Euler tour: K_n for all odd $n \geq 3$.
   Hamilton path: K_n for all $n \geq 1$.
   Hamilton cycle: K_n for all $n \geq 3$

2. (a) For what values of m and n does the complete bipartite graph $K_{m,n}$ contain an Euler tour?

   (b) Determine the length of the longest path and the longest cycle in $K_{m,n}$, for all m, n.

   Solution:

   (a) Since for connected graphs the necessary and sufficient condition is that the degree of each vertex is even, m and n must be even positive integers.

   (b) The length of the longest cycle is $2 \cdot \min\{m, n\}$: Any cycle must be even, and it must alternate vertices from the two sides. Thus it cannot be longer that twice the smaller side (and such a cycle clearly exists). By a similar reasoning, we get that if $m = n$, the longest path contains all the $2m$ vertices, so its length is $2m - 1$, and if $m \neq n$, the length of the longest path is $2 \cdot \min\{m, n\}$, starting and ending in the larger class.

3. (a) Find a graph such that every vertex has even degree but there is no Euler tour.

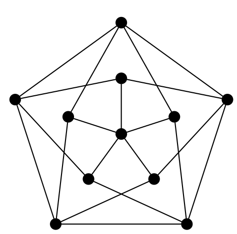
   (b) Find a disconnected graph that has an Euler tour.

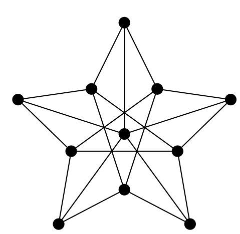
   Solution:

   (a) Take a graph that is the vertex-disjoint union of two cycles. It is not connected, so there is no Euler tour.

   (b) The empty graph on at least 2 vertices is an example. Or one can take any connected graph with an Euler tour and add some isolated vertices.

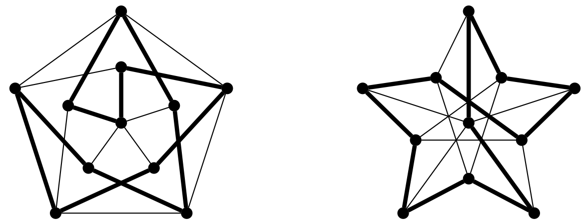
4. Determine the girth and circumference of the following graphs.

   Solution: The graph on the left has girth 4; it's easy to find a 4-cycle and see that there is no 3-cycle. It has circumference 11, since below is an 11-cycle (a Hamilton cycle).

   The graph on the right also has girth 4. It also has circumference 11, since below is an 11-cycle. Actually, one can check that the two graphs are isomorphic.



Problems

5. Let G be a connected graph that has an Euler tour. Prove or disprove the following statements.

   (a) If G is bipartite then it has an even number of edges.

   (b) If G has an even number of vertices then it has an even number of edges.

   (c) For edges e and f sharing a vertex, G has an Euler tour in which e and f appear consecutively.

   Solution.

   (a) True. The number of edges in a bipartite graph is equal to the sum of the degrees in one part. If there is an Euler tour then all degrees are even, and the sum of even numbers is also even.

   (b) False. Take a cycle of length 3 and a cycle of length 4, joined at a single vertex. It has 6 vertices and 7 edges, and it has an Euler tour. *Note: The sum of an even number of even numbers is not necessarily divisible by four!*

   (c) False. The previous example of a graph works, just select e and f touching the common vertex, but from the same cycle.

6. Show that if $k > 0$ then the edge set of any connected graph with $2k$ odd-degree vertices can be split into k trails.

   Solution: Pair up the odd-degree vertices of the graph and add an edge between any pair. To distinguish them, let us say that the new edges are blue. The resulting *multi*graph is connected with all degrees even, so it has an Euler tour. Now delete the blue edges from this tour: we get k trails that cover each edge of the original graph exactly once.

7. Prove that in any connected graph G, there is a walk that uses each edge exactly twice.

   Solution: We duplicate each edge of G in order to get the new (multi)graph G'. Since all vertices of G' have even degree by construction, G' has an Eulerian trail. This gives the desired walk.

8. Let G be a connected graph with an even number of edges such that all the degrees are even. Prove that we can color each of the edges of G red or blue in such a way that every vertex has the same number of red and blue edges touching it.

   Solution: Let T be an Eulerian tour for G. (which exists by the assumptions) We get the desired coloring by coloring the edges of T (and as a result, the edges of G) alternatively by red and blue. Note that this coloring is feasible since G has even number of edges, and on the other hand, each vertex of G is adjacent to the same number of red and blue edges.

9. Prove that the following Fleury's algorithm finds an Euler tour or an Euler trail if it is possible.

   (a) If there are 0 odd vertices, start anywhere. If there are 2 odd vertices, start at one of them.

   (b) At each step choose the next edge in the path to be one whose deletion would not disconnect the graph, unless there is no such edge, in which case it pick the remaining edge left at the current vertex.

   (c) Stop when you run out of edges.

Solution: We call the resulting walk C. Then, C has to be a trail as we clearly cannot visit an edge twice. As we know, an Euler trail only exists if exactly 0 or 2 vertices have odd degrees. If 0, then our trail must end at the starting vertice because all our vertices have even degrees. If 2, we must end at the non-starting odd-degree vertex.

Now assume C is not Eulerian and consider $G[F]$ where $F = E(G) - E(C)$. Our starting vertex s cannot be in F as the algorithm terminated when there were no incident edges to go to. F is not empty because we assumed that C was not Eulerian. We consider v_i, the last visited (during the last cycle of the algorithm) vertex in C that also is in $G[F]$. The edge $v_i v_{i+1}$ (where v_{i+1} is the next vertex visited by the algorithm after v_i during its last cycle) was chosen by the algorithm but there exists an edge f incident to v_i that was not visited by the algorithm because of how we chose v_i. For all $j \geq i + 1$, v_j is not incident to any edge of $G[F]$ by definition of i. This means $v_i v_{i+1}$ had to be a bridge (a bridge in a graph is an edge whose deletion increases the number of components) when selected, which shows that f had to be a bridge too at this point in the algorithm. As $G[F]$ is Eulerian because every degree is even, this is a contradiction as Eulerian graphs do not contain any bridges.