# Graph theory - problem set 1

September 19, 2019

### Exercises

1. Given a graph $G$ with vertex set $V = \{v_1, \ldots, v_n\}$ we define the *degree sequence* of $G$ to be the list $d(v_1), \ldots, d(v_n)$ of degrees in decreasing order. For each of the following lists, give an example of a graph with such a degree sequence or prove that no such graph exists:

   (a) $3, 3, 2, 2, 2, 1$

   (b) $6, 6, 6, 4, 4, 3, 3$

   (c) $6, 6, 6, 4, 4, 2, 2$

2. Construct two graphs that have the same degree sequence but are not isomorphic.

3. A graph is *$k$-regular* if every vertex has degree $k$. How do 1-regular graphs look like? And 2-regular graphs?

4. How many (labelled) graphs exist on a given set of $n$ vertices? How many of them contain exactly $m$ edges?

5. Prove that the number of odd-degree vertices in a graph is always even.

6. Let $G$ be a graph with minimum degree $\delta > 1$. Prove that $G$ contains a cycle of length at least $\delta + 1$.

7. Show that every graph on at least two vertices contains two vertices of equal degree.

8. Prove that at a meeting of at least 6 people, there are always 3 that mutually know each other, or 3 that mutually do not know each other.

   *Hint:* start by proving the following statement. If $G$ is a graph on at least 6 vertices, then either $G$ or its complement has a vertex of degree at least 3.

   The complement of a graph $G = (V, E)$, denoted $G^C$, is the graph with set of vertices $V$ and set of edges $E^C = \{uv \mid uv \notin E\}$.

9. What is the maximum number of edges in a bipartite graph on $n$ vertices? (Prove your answer.)