
The problem can be submitted until Mai 31, 12 :00 noon, into the box in front of MA C1 563 (Attention : There is no exercise session in this week).

Student(s) [1] :

Question 1 : *The question is worth* 5 *points.*

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 *Reserved for the corrector*

Definition : For a directed graph $G = (V, A)$ with weight function $\ell : A \to \mathbb{R}$, we define a *potential function* to be a function $d : V \to \mathbb{R}$. The *reduced weight* $\ell_d : A \to \mathbb{R}$ corresponding to d is given by :

$$\ell_d(u, v) = \ell(u, v) + d(u) - d(v)$$

for all $(u, v) \in A$.

Let $G = (V, A)$ be a directed graph with weight function $\ell : A \to \mathbb{R}$ and suppose that G has no negative cycles.

1. Let P be a shortest path between s and t with respect to ℓ. Show that for each potential function, P is also a shortest path between s and t in respect to the reduced weights.
2. Show that there exists a potential function d^+ such that all reduced weights are ≥ 0.

1. You are allowed to submit your solutions in groups of at most three students.

Sol.:

1. Let P' be a path in G. The weight of P' is the sum of the weight over all its arcs. Calculate :

$$\ell_d(P') = \sum_{(u,v)\in P'} \ell_d(u,v) = \sum_{(u,v)\in P'} [\ell(u,v) + d(u) - d(v)]$$

   Expanding this sum gives :

$$\ell_d(P') = \sum_{(u,v)\in P'} \ell(u,v) + d(u_0) - d(u_k) = \ell(P') + d(u_0) - d(u_k)$$

   Where u_0 is the first and u_k the last vertex of P'.
   Suppose that P is a shortest path between s and t in respect to ℓ. For each path P' between s and t, we have $\ell(P') \geq \ell(P)$ by definition of P. Thus we get :

$$\ell_d(P') = \ell(P') + d(s) - d(t) \geq \ell(P) + d(s) - d(t) = \ell_d(P)$$

   This implies that P is also a shortest path between s and t with respect to the reduced weight ℓ_d.

2. By splitting the graph in components, we may assume that there exists a vertex s from which each other vertex can be reached. Since the graph has no negative cycles, we can calculate the shortest paths from s to each other vertex. Define $d^+(v)$ to be the shortest distance from s to v. We want to show that $\ell_{d^+}(u,v) \geq 0$ for all arcs $(u,v) \in A$.
   Consider $(u,v) \in A$, we know that $d^+(u)$ is the shortest distance from s to u. Since (u,v) is an arc in G, the shortest distance between s and v is $\leq d^+(u) + \ell(u,v)$. This gives the inequality :

$$d^+(v) \leq d^+(u) + \ell(u,v)$$

   Rearranging gives :

$$0 \leq \ell(u,v) + d^+(u) - d^+(v) = \ell_{d^+}(u,v)$$

   Where the equation is by definition of ℓ_{d^+}.