École Polytechnique Fédérale de Lausanne
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Mai 24, 2019

Discrete Optimization (Spring 2019)

Assignment 12

Problem 1

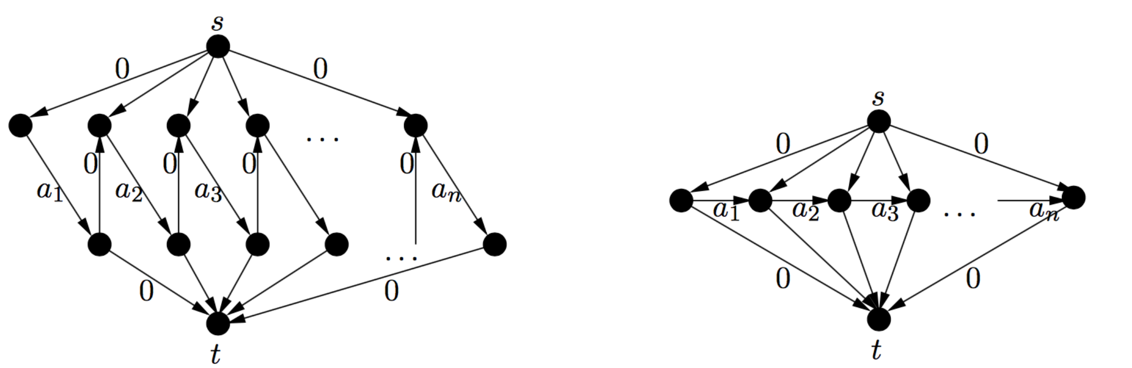
Given n numbers $a_1, \ldots, a_n$ find indices i and j, $1 \leq i \leq j \leq n$, such that $\sum_{k=i}^{j} a_k$ is minimized. We will develop two algorithms for this problem that run in linear time, *i.e.*, the number of (arithmetic) operations is linear in n.

(a) Solve the problem using Bellman-Ford as a subroutine. In particular, construct a graph such that a shortest path in this graph yields the optimal solution to the above problem. Show that the graph can be generated in linear time and that Bellman-Ford can be implemented to run in linear time on this graph.

(b) Define $d(j) = \min_{1 \leq i \leq j} \sum_{k=i}^{j} a_k$. Conclude that the above problem is equivalent to computing $\min_{1 \leq j \leq n} d(j)$. Show how this can be done in linear time.

Solution:

(a) Consider the graph in Figure 1(a). The shortest path between s and t gives the respective



(a) General instance

(b) Instances with at least one negative number.

result. Clearly, this graph is acyclic and it has $O(n)$ edges so by Problem 4, one can compute the shortest $s - t$ path in $O(n)$ operations.

If there is at least one negative number, we can also consider the simpler graph in Figure 1(b). Note that instances with only non-negative numbers are not very interesting. Then, the solution is $i = j$ where a_i is the smallest number.

(b) It is easy to see that $d(j+1) = \min\{d(j) + a_{j+1}, a_{j+1}\}$. Setting $d(1) = a_1$ we can thus compute $d(2), \ldots, d(n)$ subsequently. Each successor computation takes constant time, hence the total computation takes $O(n)$ time.

The optimal pair (i, j) sums the numbers from a_i to a_j. Hence, this sequence ends at j and the value of $d(j) = \sum_{k=i}^{j} a_k$. Thus, after computing the values of d we can select the minimum value of the values in d which yields the optimal value. Again, this can be done in linear time.

Problem 2

Due to the decentralized nature of the global currency market, it might be the case that an individual or an organized group makes a large profit without risk. Arbitrage is a phenomenon that refers

to cases when it is possible to convert one unit of a currency into more than one unit of the same currency by using discrepancy in exchange rates. For example, consider the case that 1 CHF buys 60 RUB, 1 RUB buys 0.019 USD and 1 USD buys 0.93 CHF. This means that a trader can transform 1 CHF into $60 \cdot 0.019 \cdot 0.93 = 1.0602$ CHF gaining a profit of 6.02%.
Given a list of currencies $r_1, \dots, r_n$ and a matrix $E \in \mathbb{R}_{>0}^{n\times n}$ where $E_{i,j}$ specifies the exchange rate between currencies r_i and r_j, design a polynomial time algorithm to test if there is a possibility of arbitrage. While modelling the problem, bear in mind that testing if a weighted directed graph has a negative cycle can be done in polynomial time.

Solution:
Observe that in the case of arbitrage one makes a circular sequence of trades $t \in T$. The profit has been obtained in the case when the product of the corresponding exchange rates

$$\prod_{t\in T} E_{i(t),j(t)} > 1 \iff \prod_{t\in T} \frac{1}{E_{i(t),j(t)}} < 1 \iff \sum_{t\in T} \log\left(\frac{1}{E_{i(t),j(t)}}\right) < 0,$$

where $i(t)$ indicates the index of the currency sold in transaction t and $j(t)$ is the index of the currency bought.
The above statement suggests the following natural construction. Let $G = (V, E)$ be a complete directed graph with the set of nodes $V = r_1, \dots, r_n$. For each arc $(r_i, r_j) \in E$ set the weight $w_{i,j} = \log\left(\frac{1}{E_{i,j}}\right)$. The (possibly empty) set of arbitrages is in one-to-one correspondence with the set of negative-weight directed cycles of on G. Existence of a negative cycle can be detected by running n iterations of the Bellman-Ford algorithm. Such a cycle exists if and only if the distance matrix gets updated in the $n-th$ iteration as mentioned in class. It can be reconstructed by using the list of predecessors maintained by the algorithm.

Problem 3
Let $D = (V, A)$ be a directed graph, $w : A \to \mathbb{R}$ be arc weights and $s \in V$. Suppose that there exists a path from s to each other node of V.
Consider the folowing linear program:

$$\begin{aligned} \max \quad & \sum_{v\in V\setminus\{s\}} x_v \\ \text{s.t.} \quad & x_v - x_u \le w(u,v), \quad \forall (u,v) \in A \\ & x_s \le 0. \end{aligned} \tag{1}$$

Show the following:

a) This LP is feasible if and only if D has no negative cycle;

b) If D has no negative cycle, then (1) has a unique optimal solution.

Solution:
For each $v \in V$, denote with $d(s, v)$ the length of the shortest path from s to v in D, subject to w.

a) ($\Leftarrow$) If there are no negative cycles in D, then we have for each $(u, v) \in A$:

$$d(s,v) \le d(s,u) + w(u,v) \Leftrightarrow d(s,v) - d(s,u) \le w(u,v).$$

Thus, the assignment $x^*_v = d(s,v),\ \forall v \in V$ with $x^*_s = 0$ is feasible for (1).

($\Rightarrow$) Let $\bar{x}$ be a feasible solution to (1) and let C be a directed cycle in D, denote its length with $w(C)$. One has:

$$w(C) = \sum_{(u,v)\in C} w(u,v) \ge \sum_{(u,v)\in C} (\bar{x}_v - \bar{x}_u) = 0.$$

The last equality comes from the fact that every vertex contained in C arises in the summation exactly twice, once with the positive and once with the negative sign.

b) Consider an arbitrary vertex $v \in V$ and let $s = u_0 \to u_1 \to \cdots \to u_k = v$ be a shortest length path from s to v, subject to w. By adding up the constraints $x_{u_1} - x_s \leq w(s, u_1)$ and $x_s \leq 0$ we obtain that $x_{u_1} \leq w(s, u_1)$ is valid for (1). Further adding $x_{u_2} - x_{u_1} \leq w(u_1, u_2)$ gives $x_{u_2} \leq w(s, u_1) + w(u_1, u_2)$. If we continue the process in the same manner, then we have that

$$\underbrace{x_v}_{x_{u_k}} \leq \sum_{i=1}^{k} w(u_{i-1}, u_i) = d(s, v)$$

For each $v \in V$, this inequality has to be satisfied by every feasible solution of (1). Thus, the maximum of $\sum_{v \in V \setminus \{s\}} x_v$ is uniquely attained by the solution

$$x_v^* = d(s, v), \quad \forall v \in V \setminus \{s\}.$$

Note that, if there is no negative cycle in D, there has to be an (outgoing) arc $(s, v) \in A$ such that $d(s, v) = w(s, v)$. Therefore, $x_s^* \geq x_v^* - w(s, v) = 0$, and by $x_s^* \leq 0$ one gets $x_s^* = 0$. As seen in part (a), the solution x^* is feasible for (1).

Problem 4

Design an algorithm that, a directed graph $G = (V, A)$, finds the number of shortest paths from s to t in time $O(|V| + |A|)$.

Solution:

We will modify the breadth-first search algorithm in a way that it also counts the number of shortest paths from s to any other vertex. This is done by inheriting the number of paths from the predecessor and updating this number if other paths of the same length are found.

More formally: keep the initialization for the queue Q, the distance array D and the predecessor array π. Further, add an array p storing the number of paths to each node and initialize $p = [1, 0, \ldots, 0]$.

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while Q ≠ ∅ do
    u := head(Q)
    for each v ∈ δ⁺(u) do
        if D[v] = ∞ then
            π[v] := u
            D[v] := D[u] + 1
            p[v] := p[u]
            enqueue(Q, v)
        else if D[v] = D[u] + 1 then
            p[v] := p[v] + p[u]
    dequeue(Q)
```

As seen in class, the algorithm finds all shortest paths from s. Since each shortest path from s to v goes through exactly one vertex in $\delta^-(v)$, the number of shortest paths from s to v is exactly the sum of the number of paths of length $D[v] - 1$ from s to a vertex in $\delta^-(v)$. Note that there will never be a path of length $< D[v] - 1$ from s to a vertex in $\delta^-(v)$, otherwise $D[v]$ would not be the length of the shortest path from s to v. Thus for each $u \in \delta^-(v)$, the number of paths of length $D[v] - 1$ is either 0 or $p[v]$.

Since we add only a constant number of calculations in each iteration to the breath-first search algorithm the running time remains $O(|V| + |A|)$.

Problem 5

A 2-matching in a graph is a collection of disjoint cycles that covers all the vertices. Show that a 2-matching can be computed in polynomial time, if such one exists. Note that it is allowed to pick an edge twice in a 2-matching, i.e., one can have a 2-cycle.

Hint: One may reduce the problem to finding a perfect matching in a bipartite graph.

Solution:
Given the initial graph $G(V, E)$, construct a bipartite graph $G' = (V \cup V', E')$, where V' is a copy of V and E' has the edges $\{u, v'\}$ and $\{u', v\}$ for each edge $\{u, v\} \in E$.
Now, every 2-matching $M_{(2)}$ in G corresponds to a perfect matching M' in G'. For every cycle $v_1, \ldots, v_k$ in $M_{(2)}$ put the edges $(v_1, v_2'), \ldots, (v_{k-1}, v_k'), (v_k, v_1')$ in M'. Clearly, M' matches all the vertices $v_1, \ldots, v_k, v_1', \ldots, v_k'$. Doing this for all cycles in matching $M_{(2)}$ of G yields the corresponding perfect matching M' of G'.
Conversely, if there is a perfect matching M' in G', we can construct a 2-matching $M_{(2)}$ in G. For each of edges $\{u, v'\}$ and $\{u', v\}$ in M', we add the edge $\{u, v\}$ to $M_{(2)}$. A perfect matching in G' can be computed, or it can be detected that there is no such matching, in polynomial time.