Discrete Optimization
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Spring 2019
18.04.2019

Midterm Exam

Last name: First name:

Exercise:	1	2	3	4	5	6	7	Σ	
max points:	10	10	10	10	10	10	10	60	
achieved points:									
chosen exercises:									

Remark - read me first:
The exercises in this practice exam are mainly taken from previous exams on discrete optimization. Furthermore, note that these exercises only cover a subset of the topics of the class. We advise you to also refresh your memory on all the other subjects.
Good luck with your preparations!

Correction: If you would like to get a feedback for the midterm, please submit your exercise sheet at the next exercise session at latest. As in the final exam, you can choose 6 out of 7 exercises to work on.

Regulations for the final exam:

- Duration: 3 hours.
- Check whether the exam is complete: It should have ?? pages (Exercises 1–7).
- Write your name on the title page. Put your CAMIPRO card on your table.
- Use neither pencil nor red colored pen!
- Solutions have to be written below the exercises. Solutions must be comprehensible.
- In case of lack of space, you can ask for additional paper from the exam supervision. Please put your name on each additional sheet and indicate which exercise it belongs to.

No additional aids are allowed to the exam

Exercise 1 (Problem modelling)

A factory produces two different products. To create one unit of product 1, it needs one unit of raw material A and one unit of raw material B. To create one unit of product 2, it needs one unit of raw material B and two units of raw material C. Raw material B needs preprocessing before it can be used, which takes one minute per unit. At most 20 hours of time is available per day for the preprocessing. Raw materials of capacity at most 1200 can be delivered to the factory per day. One unit of raw material A, B and C has size 4, 3 and 2 respectively.
At most 130 units of the first and 100 units of the second product can be sold per day. The first product sells for 6 CHF per unit and the second one for 9 CHF per unit.
Formulate the problem of maximizing turnover as a linear program in two variables.

Solution 1

We introduce 5 variables. Variables x and y model the amount of units of product one and two respectively that should be created. Variables a, b and c model the amount of raw material A, B and C needed.
One unit of product one sells for 6 CHF and one unit of product two sells for 9 CHF per unit. Thus the turnover is $6x+9y$. This is our objective function.
To create one unit of product one we need one unit of A. Thus $x \leq a$ has to hold for each feasible solution. For each unit of product one and two we need one unit of B. Thus $x+y \leq b$ must hold. Similarly we get the constraint $2y \leq c$.
Raw material B needs preprocessing that takes 1 minute per unit. At most 20 hours are available for preprocessing. This limits the maximum amount of b: $b \leq 60 \cdot 20 = 1200$.
The total size of raw materials we can have per day is limited by 1200. One unit of raw material A, B and C has size 4, 3 and 2 respectively. This gives the constraint $4a+3b+2c \leq 1200$.
At most 130 units of product 1 and 100 units of product two can be sold per day, thus $x \leq 130$ and $y \leq 100$ must hold. Finally all variables must be nonnegative since we cannot produce a negative amount of products or have a negative stock of raw material: $x,y,a,b,c \geq 0$.
This yields the following linear program:

$$\begin{aligned}
\max \quad & 6x+9y && \\
\text{subject to} \quad & x && \leq a \\
& x+y && \leq b \\
& 2y && \leq c \\
& b && \leq 1200 \\
& 4a+3b+2c && \leq 1200 \\
& x && \leq 130 \\
& y && \leq 100 \\
& x,y,a,b,c \geq 0 &&
\end{aligned} \tag{1}$$

This model has 5 variables, and we will now eliminate three of them. Consider an optimal solution x^*,y^*,a^*,b^*,c^* of the linear program (1). Observe that if we decrease a^*, b^* and c^* such that the first three constraints are satisfies with equality, the solution stays feasible and the objective value does not change. Thus this modified solution is optimal as well.
This justifies that we consider the modified LP instead where the fist three inequalities are replaced by equalities, i.e.

$$
\begin{array}{rll}
\max & 6x+9y & \\
\text{subject to} & x & = a \\
& x+y & = b \\
& 2y & = c \\
& b & \leq 1200 \\
& 4a+3b+2c & \leq 1200 \\
& x & \leq 130 \\
& y & \leq 100 \\
& x,y,a,b,c & \geq 0
\end{array}
$$

By substituting a, b and c in the remaining inequalities we get a linear program with 2 variables

$$
\begin{array}{rll}
\max & 6x+9y & \\
\text{subject to} & x+y & \leq 1200 \\
& 4x+3(x+y)+2\cdot 2y & \leq 1200 \\
& x & \leq 130 \\
& y & \leq 100 \\
& x,y & \geq 0
\end{array}
$$

where we can remove the first inequality since the second one is strictly stronger:

$$
\begin{array}{rll}
\max & 6x+9y & \\
\text{subject to} & 7x+7y & \leq 1200 \\
& x & \leq 130 \\
& y & \leq 100 \\
& x,y & \geq 0
\end{array}
$$

Exercice 2 (Polyhedral theory)

Let $x \in P$, P a polyhedron. Show that x is an extreme point of P if and only if it cannot be written as a convex combination of other points in P.

Solution 2

If x is an extreme point, then by definition, $\{x\} = P \cap \{x \in \mathbb{R}^n \mid a^T x = \beta\}$. So if x could be written as a convex combination of other points, i.e. $x = \sum_{i=1}^m \lambda_i v_i$ and $\sum_{i=1}^m \lambda_i = 1$, $\lambda_i \geq 0$ for $i \in [m]$, then we arrive at the following contradiction:

$$a^T x = a^T \left(\sum_{i=1}^m \lambda_i v_i \right) = \sum_{i=1}^m \lambda_i a^T v_i < \sum_{i=1}^m \lambda_i \beta = \beta$$

For the other direction, we see that in x, (at least) n valid inequalities defining the polyhedron have to be tight (else, there exists some v such that $x \pm \lambda v \in P$ for some small λ and some vector v). Let these valid inequalities be $a_1 x \leq \beta_1, \cdots, a_n x \leq \beta_n$. So the the objective function $\sum_{i=1}^n a_i$ has objective value $\sum_{i=1}^n \beta_i$ at x and is strictly smaller for all other points.

Exercise 3 (Simplex phase II)

Consider the following LP:

$$
\begin{array}{lrcrcrcl}
\max & y_1 & + & 2y_2 & + & 3y_3 & & \\
 & -y_1 & + & 4y_2 & + & 2y_3 & \leq & 5 \\
 & 2y_1 & - & 6y_2 & - & y_3 & \leq & 2 \\
 & 2y_1 & - & 3y_2 & + & 4y_3 & \leq & 1 \\
 & -y_1 & & & & & \leq & 0 \\
 & -y_2 & & & & & \leq & 0 \\
 & -y_3 & & & & & \leq & 0
\end{array}
$$

Solve the linear program using the simplex algorithm with Bland's pivoting rule.
Start with the basis $B = \{4,5,6\}$ and the corresponding vertex $(0,0,0)^T$.
For each iteration of the simplex algorithm, indicate the current basis and the corresponding vertex (basic feasible solution).

At the end provide the optimal vertex, its objective function value and the certificate of optimality.

The inverse matrices of all feasible bases are:

$$
B = \{1,3,4\} \Longrightarrow A_B^{-1} = \begin{bmatrix} 0 & 0 & -1 \\ 2/11 & -1/11 & -4/11 \\ 3/22 & 2/11 & 5/22 \end{bmatrix} \quad B = \{1,3,6\} \Longrightarrow A_B^{-1} = \begin{bmatrix} 3/5 & 4/5 & 22/5 \\ 2/5 & 1/5 & 8/5 \\ 0 & 0 & -1 \end{bmatrix}
$$

$$
B = \{1,4,6\} \Longrightarrow A_B^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 1/4 & -1/4 & 1/2 \\ 0 & 0 & -1 \end{bmatrix} \quad B = \{3,4,5\} \Longrightarrow A_B^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1/4 & 1/2 & -3/4 \end{bmatrix}
$$

$$
B = \{3,5,6\} \Longrightarrow A_B^{-1} = \begin{bmatrix} 1/2 & -3/2 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad B = \{4,5,6\} \Longrightarrow A_B^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.
$$

Solution 3

iter	basis	vertex
0	$\{4,5,6\}$	$(0,0,0)^T$
1	$\{3,5,6\}$	$(1/2,0,0)^T$
2	$\{1,3,6\}$	$(19/5,11/5,0)^T$

The optimum is reached at the vertex $(19/5,11/5,0)^T$, the corresponding objective value is $41/5$ and the certificate is the vector

$$
\lambda^T = (7/5, 0, 6/5, 0, 0, 23/5),
$$

.

Exercise 4 (Strong duality)
Consider the following linear program:

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 6 \\ & x_1 + 2x_2 \leq 8 \\ & 3x_1 + 4x_2 \leq 22 \\ & x_1 + 5x_2 \leq 23 \end{aligned}$$

Show that $(4/3, 10/3)$ is an optimal solution by using the notion of optimal basis or duality.

Solution 4
The assignment $(4/3, 10/3)$ has the objective function value of $14/3$. In order to prove that it is optimal (via strong duality), we are going to form the dual LP, and find a feasible solution to the dual that achieves the same objective value. The dual is:

$$\begin{aligned} \min \quad & 6y_1 + 8y_2 + 22y_3 + 23y_4 \\ \text{subject to} \quad & 2y_1 + y_2 + 3y_3 + y_4 = 1 \quad (1) \\ & y_1 + 2y_2 + 4y_3 + 5y_4 = 1 \quad (2) \\ & y_1, y_2, y_3, y_4 \geq 0 \end{aligned}$$

Thus, we are looking for a feasible dual solution such that $6y_1 + 8y_2 + 22y_3 + 23y_4 = 14/3$. By using Gaussian elimination on this constraint combined with (1) and (2) we get:

$$-4y_2 - 2y_3 - 7y_4 = -4/3,$$

$$-3y_2 - 5y_3 - 9y_4 = -1$$

and further

$$14/3y_3 + 5y_4 = 0.$$

Since $y_1, y_2, y_3, y_4 \geq 0$, we have that $y_3 = y_4 = 0$ and then $y_1 = y_2 = 1/3$. This is the desired feasible dual solution coinciding with the primal solution $(4/3, 10/3)$, proving the optimality of the latter.

Exercise 5 (Algorithms, O-notation)

Let $a, e, k \in \mathbb{N}$ be three given natural numbers.

a) Argue that a^{2^k} can be computed using $\Theta(k)$ multiplications.

b) How many bits (Θ-notation) has a^{2^k}?

c) Let $(e_0, \ldots, e_\ell)$ be the bit-representation of e, i.e., $e = \sum i = 0^\ell e_i 2^i$ with $e_i \in \{0,1\}$ for $i = 0, \ldots, l$. Complete the following algorithm by replacing each occurrence of three question marks (???) so that it computes a^e using $O(\ell)$ many arithmetic operations.

```
E = 1
S = a

For (i=0 to l)
  if (e_i == 1)
    E = E* ???

  S = ???

return ???
```

d) Show that for given $a, e, N \in \mathbb{N}$ one can compute $a^e \pmod N$ in time polynomial in the binary encoding length of a, e and N.

e) Let $a, b, c, N \in \mathbb{N}$ be given and suppose that N is a prime number. Show that $a^{b^c} \pmod N$ can be computed in polynomial time in the binary encoding length of a, b, c and N. You may use Fermat's little theorem: $a^N \equiv a \pmod N$.

Solution 5

a) Start with $a(0) = a$, and compute $a(i) = a(i-1) \cdot a(i-1)$ for $i \in [k]$. By induction

$$a(k) = a^{2^{k-1}} \cdot a^{2^{k-1}} = a^{2^k}.$$

Thus, a^{2^k} has been obtained by performing k multiplications.

b) It has $\log(a^{2^k}) = \Theta(2^k \log a)$ bits.

c) One has `E = E*S`, `S = S*S` and `return E`.

d) We use the same algorithm as in part c), but do operations modulo N, in particular we replace `E = E*S` with `E = E*S mod N` and `S = S*S` with `S = S*S mod N`. Observe that the modulo can be obtained by performing division with remainder. The correctness of the algorithm follows from the fact that for any $a, b \in \mathbb{N}$, $ab \equiv (a \pmod N)(b \pmod N) \pmod N$. The algorithm performs $O(\ell)$ steps, where ℓ is the size of e, and in each step we perform division with remainder on $E * S$ (or $S * S$) and N: this takes time polynomial in the binary encoding length (size) of the input, in particular of a, e, N (notice that the value of $A \pmod N$ has size at most the size of N, irrespectively of the size of A). Hence the total number of operations performed is polynomial in the size of a, e, N.

e) Using the hint, we have that $a^{b^c} \equiv a^{b^c \pmod{N-1}} \pmod N$. Hence we first apply the algorithm from part d) to obtain $b^c \pmod{N-1}$ in polynomial time in the size of b, c, N. Notice that the size of this number, which we denote by e, is bounded by the size of N only. Now, we apply again the algorithm from part d) to obtain $a^e \pmod N$, in time polynomial in the size of a, N.

Exercise 6 (Polyhedral theory)

Let

$$\max\{c^T x \ : \ Ax \leq b\} \tag{2}$$

be a feasible linear program. Show that (2) is bounded if and only if the program

$$\max\{c^T x \ : \ Ax \leq \mathbf{0}, \ c^T x \leq 1\} \tag{3}$$

has optimal value equal to 0.

Solution 6 The linear program (2) is unbounded if and only if there is a half line contained in $\{Ax \leq b\}$, i.e. there are $x, \ y \in \mathbb{R}^n$ such that $Ay \leq b, \ A(\lambda x + y) \leq b$ for any $\lambda \geq 0$, and $c^T x > 0$. This happens if and only if there is an x such that $Ax \leq 0$ and $c^T x > 0$, which is equivalent to saying that (3) has optimum value greater than 0 (the point x can be scaled such that $c^T x \leq 1$).

Exercise 7 (Integral polytope)
Let P be the following n-dimensional polytope: $P = \{x \in \mathbb{R}^n : \ -1 \leq x_i \leq 1, \ i = 1, \dots, n\}$. Describe the set of all vertices of P and prove that it is correct.

Clearly P is the cube and its vertices are of the form $(\pm 1, \dots, \pm 1)$. One way to see it is using the equivalence between vertices and basic feasible solutions: each vertex must satisfy with equality n linearly independent constraints. There are in total $2n$ constraints and for any $i : x_i = 1, \ x_i = -1$ cannot be both satisfied, hence each vertex v will satisfy a set of constraints $x_i = 1$ for $i \in I_1 \subset [n]$, $x_j = -1$ for $j \in [n] \setminus I_1$ (note that the corresponding inequalities are linearly independent), so v has the form $(\pm 1, \dots, \pm 1)$. Conversely, any point of this form will be a vertex of P since it is in P and satisfies n linearly independent constraints with equality.