Discrete optimization
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EPFL

The problem can be submitted until Mai 17, 12 :00 noon, into the box in front of MA C1 563 or during the exercise session.

Student(s) [1] :

Question 1 : *The question is worth* 5 *points.*

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 *Reserved for the corrector*

For a polyhedron P with vertices V and edges E, we can consider the graph $G = (V, E)$. The combinatorial diameter of this graph, $diam(P)$, is the minimum number of edges needed in order to reach any vertex by a path from any other vertex. Obtaining a good bound on $diam(P)$ is a very difficult and unsolved problem, see for instance the Hirsch conjecture. But for the special case where P is bounded and the vertices of P are in $\{0, 1\}^n$, Naddef has shown (and you will too...) that $diam(P) \leq n$. It might be useful to follow these steps :

1. Show the theorem for $n = 1$. (It might be useful to think about why it also works for $n = 2$).
2. Show that $P_1 := \{x \in \mathbb{R}^n \mid e_1^T x = 1\} \cap P$ corresponds to a polyhedron in dimension $n - 1$ with vertices in $\{0, 1\}^{n-1}$.
3. Show that set of vertices and edges of P_1, V_1 and E_1, is a subset of the vertices of P, i.e. $V_1 \subseteq V$ and $E_1 \subseteq E$.
4. If we are at vertex $v \in P$ with $v_1 = 0$ and there exists another vertex $w \in P$ with $w_1 = 1$, show that there must be a neighbour $\bar{v}$ of v (i.e. there is an edge between v and $\bar{v}$) such that $\bar{v}_1 = 1$.
5. Conclude.

We will use the following notation : for a vertex v, let c_v be the objective function that is (strictly) optimal for v, i.e., $\{x \in \mathbb{R}^n \mid c_v^T x = \beta\} \cap P = \{v\}$ and for all $p \in P/\{v\}$, $c_v^T p < \beta$. The existence of such a objective function is guaranteed by the definition of an extreme point. Similarly, recall the characterization of edges shown in homework 4 : For $u, v \in P$, uv is an edge of the polytope if and only if there exists an objective function c_{uv} such that c_{uv} is strictly optimal for only vertices u and v, i.e. $\{x \in \mathbb{R}^n \mid c_{uv}^T x = \gamma\} \cap P = \text{seg}(u, v)$ and for all $p \in P \setminus \text{seg}(u, v)$, $c_{uv}^T p < \gamma$.

For $n = 1$, there are 4 possibilities, either P is empty, consists of only one vertex $\{0\}$ or $\{1\}$, or is the line segment $[0, 1]$ (in this case there is an edge from $\{0\}$ to $\{1\}$ since the inequality $\{x \in \mathbb{R}^1 \mid 0^T x = 0\}$ is a valid inequality for $[0, 1]$ and contains exactly the line segment...). In any case, the combinatorial diameter for $n = 1$ is at most 1.

Let us first explain why steps 1 to 4 imply the result. Let v and w two arbitrary vertices and we need to show there is a path of length at most n connecting them. Suppose

1. You are allowed to submit your solutions in groups of at most three students.

that $v_1 = 0$ and $w_1 = 1$. In that case, by 4, there is a neighbour of w_1 with first coordinate 0 (so we need to take 1 edge to move from w_1 to some other vertex with first coordinate 0). So assume $v_1 = w_1 = 0$ (similar for $v_1 = w_1 = 1$). In that case, we intersect the polytope with the hyperplane $\{x \in \mathbb{R}^n \mid e_1 x = 0\}$, this gives us the polytope P_0. Since all vertices have the same first coordinate, we can just delete the first coordinate from P_0 and think of it as being a subset of $\{0,1\}^{n-1}$. By step 4, this correspondance does not "create" new vertices or new edges (implying a short path), or vertices or edges of P_0 are no vertices resp. edges anymore... Assuming all this, a path between v and w in P_0 (we do not leave P_0 anymore!), corresponds to a path (of the same length) in the original polytope P. This allows us to use recursion, since $P_0 \subseteq \{0,1\}^{n-1}$: Since it takes at most one step so that the first coordinate of both v and w agree, we have

$$diam(P) \leq 1 + diam(P_0) \leq 1 + n - 1 = n$$

First we show step 3 : We will show that if we intersect some polyhedron P with a valid inequality $H = \{x \in \mathbb{R}^n \mid a^T x = \beta\}$, then v is a vertex of $P \cap H$ if and only if v is a vertex of P and $a^T v = \beta$. If v is a vertex of $P \cap H$, then there exists a objective function $\bar{c}_v$ maximal only for v, i.e. $\{x \in \mathbb{R}^n \mid \bar{c}_v^T x = \gamma\} \cap P \cap H = \{v\}$ and for all $p \in P \cap H / \{v\}$, $\bar{c}_v^T p < \gamma$. Our goal is to construct a objective function c_v that is strictly optimal for v, but for all points $p \in P$ (not only for $P \cap H$!). Since an objective function always reaches its optimal value only in vertices, all vertices w of P not in H are such that $a^T w \leq \beta - \varepsilon$ for some $\varepsilon > 0$. We pick the objective function $c = a + \lambda c_v$ for some $\lambda > 0$. All vertices w of P not in H, for sufficiently small λ, are such that $c^T w < c^T v$. For all vertices w in $P \cap H$, we have $(a + \lambda c_v)^T w = a^T w + \lambda c_v^T w < a^T v + \lambda c_v^T v$ since $\lambda > 0$. Conversely, if v is a vertex of P and $v \in H$, then c_v is also strictly optimal for v among all points in $P \cap H$. We can do the same trick for the edges : If seg(uv) is an edge of $P \cap H$ with corresponding objective function $\bar{c}_{uv}$, then for sufficiently small $\lambda > 0$, $c_{uv} = a + \lambda \bar{c}_{uv}$ is strictly optimal for seg(uv) considering all points of P. As for the vertices, the converse is also trivially true.

We show step 2 : To pass from a polyhedron in $\mathbb{R}^n$ to a polyhedron in $\mathbb{R}^{n-1}$, for both $P_0 := \{x \in \mathbb{R}^n \mid e_1^T = 0\}$ and P_1 we leave away the first coordinate. If c_v is the objective function "defining" some vertex v, then $\bar{c}_v$ (leaving away the first coordinate of c_v) is still strictly optimal for $\bar{v}$ for P_0 or P_1 respectively (since all points in P_0 or P_1 share the same first coordinate). To go back to $\mathbb{R}^n$, we add 0 or 1 as a first coordinate to $\bar{v}$ and to its corresponding objective function $\bar{c}_v$.

We show step 4 : Paraphrased it means the following : if for some vertex v we have $e_1^T v = 0$ and $\max\{e_1^T x \mid x \in P\} = 1$, there is a neighbour w of v such that $e_1 w = 1$. Pick c_v strictly optimal for v. and consider the objective function $c(\lambda) = c_v + \lambda e_1$. For $\lambda > 0$ but sufficiently small, $c(\lambda)$ is still strictly optimal for v. But since there is a point in the polyhedron with first coordinate 1, if $\lambda \to \infty$, $c(\lambda)$ is not optimal for v anymore. So let $\bar{\lambda}$ be the supremum over all λ such that $c(\lambda)$ strictly optimal for v. Then for some $w \in P$ (there might more than one!), we have

$$(c_v + \bar{\lambda} e_1)^T v = (c_v + \bar{\lambda} e_1)^T w$$

Any such w must have first coordinate 1 - else $(c_v + \bar{\lambda} e_1)^T w = c_v^T w < c_v^T v = (c_v + \bar{\lambda} e_1)^T v$. We must resolve the issue that there might be more than one such w, this issue arises if there exist two distinct vertices $w_1, w_2 \in P$ such that $c_v(w_1 - w_2) = 0$. We show that we can assume that this situation cannot arise by showing that we can replace c_v by some other objective function d_v, strictly optimal for v only, such that for all distinct vertices $w_1, w_2 \in P$, we have

$$d_v^T w_1 \neq d_v^T w_2$$

Since we have only finitely many vertices, up to considering a multiple of c_v, we may assume either $c_v^T(w_1 - w_2) = 0$ (1) or $|c_v^T(w_1 - w_2)| \geq 1$ (2) for all vertices $w_1, w_2 \in P$. With the same reasoning as before, we see that $c_v + \varepsilon(\frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{2^n})$, for some sufficiently small $1 > \varepsilon > 0$, is strictly optimal for v. Furthermore, since all vertices of P are in $\{0, 1\}^n$, for w_1, w_2 distinct vertices of P, if (1) holds, then

$$(\frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{2^n})^T (w_1 - w_2) \neq 0$$

and, if (2) holds,

$$|(c_v^T + \varepsilon(\frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{2^n}))^T (w_1 - w_2)| \geq |c_v^T(w_1 - w_2)| - |\varepsilon(\frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{2^n}))^T (w_1 - w_2)| > 0$$

Setting $c_v + \varepsilon(\frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{2^n})$ as the strictly optimal objective function d_v for v, we are done.