Discrete optimization | Thursday, Mai 2, 2019
Prof. F. Eisenbrand | EPFL

The problem can be submitted until Mai 10, 12 :00 noon, either at the exercise session or into the box in front of MA C1 563.

Student(s) [1] :

Question 1 : *The question is worth* 5 *points.*

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 *Reserved for the corrector*

Given a graph G, a perfect matching of G is a matching which covers all the vertices (equivalently, a matching of cardinality $|V|/2$).
Suppose you are given an oracle that, given a graph G, tells you whether G has a perfect matching or not. Show how to use this oracle to determine the maximum cardinality matching of a graph $G(V, E)$. The total number of calls to the oracle (to find the cardinality of the maximum matching, and then to find the matching itself) should be at most $|V| + |E|$.

1. You are allowed to submit your solutions in groups of at most three students.

Sol.:

For $k = 0, \ldots, n = |V|$, let $G + k$ be the graph obtained by adding k dummy vertices to G which are joint to all vertices of G. Since a matching covers an even number of nodes, in what follows we only consider values of k of the same parity as n. Notice that $G + k$ has a perfect matching if and only if G has a matching of size $\frac{n-k}{2}$. We call the oracle on $G + k$, starting with $k = 0$ (or 1, depending on the parity of n) and increasing it, until $G + k$ has a perfect matching. For the minimum such k, we know that there is a matching M of size $\frac{n-k}{2}$ and it has maximum cardinality.

To find such a matching, we remove one edge $e \in E$ from $G + k$ at time and we ask the oracle if this graph has a perfect matching :

— If it doesn't, then e is in each remaining maximum cardinality matching of G. We remember e and continue with the graph obtained by deleting e and its endpoints.

— If it does, then there exists a maximum cardinality matching of G which does not contain e. We continue with the graph obtained by deleting just e.

In this way, we will find a maximum cardinality matching of G and the total number of calls is at most $k + |E| \leq n + |E|$.