The problem can be submitted until May 3, 12 :00 noon, either at the exercise session or into the box in front of MA C1 563.

Student(s) [1] :

Question 1 : *The question is worth 5 points.*

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 *Reserved for the corrector*

Given $c \in \mathbb{R}^n_+$, $a \in \mathbb{R}^n_+$ and $\gamma \in \mathbb{R}_+$, design an algorithm which, in $O(n \log n)$ operations, computes the optimal solution x^* to the following linear program :

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & a^T x \leq \gamma, \\ & 0 \leq x_i \leq 1, \quad \forall i \in [n]. \end{aligned}$$

You may assume that a set of n real numbers can be sorted in time $O(n \log n)$ and that each arithmetic operation takes constant time. It is important to prove that the solution returned by the designed algorithm is optimal.

Sol.: *We assume that the bove LP is feasible, which is easy to check. Now, sort items $i \in [n]$ monotonically decreasing according to the ratio $\frac{a_i}{c_i}$ in time $O(n \log n)$, thus re-index them so that*

$$\frac{c_1}{a_1} \geq \frac{c_2}{a_2} \geq \cdots \geq \frac{c_n}{a_n}.$$

Let k be the maximum index in $[n]$ so that

$$\sum_{i=1}^{k} a_i \leq \gamma.$$

Define a solution $\bar{x}$ with $\bar{x}_i = 1$ for $i \leq k$, $\bar{x}_{k+1} = \frac{\gamma - \sum_{i=1}^k a_i}{a_{k+1}}$ and $\bar{x}_i = 0$ for $i > k+1$. We will prove that $\bar{x}$ is an optimal solution to the above LP.

Let x^* be an arbitrary feasible solution, we would like to show that $c^T x^* \leq c^T \bar{x}$, i.e.,

$$\sum_{i>k} c_i x_i^* \leq \sum_{i \leq k} c_i (\bar{x}_i - x_i^*) + \sum_{i>k} c_i \bar{x}_i.$$

We have that :

$$\sum_{i>k} c_i x_i^* = \sum_{i>k} \frac{c_i}{a_i} a_i x_i^* \leq \sum_{i>k} \frac{c_{k+1}}{a_{k+1}} a_i x_i^* = \frac{c_{k+1}}{a_{k+1}} \sum_{i>k} a_i x_i^*.$$

1. You are allowed to submit your solutions in groups of at most three students.

Similarly, we obtain that :

$$\begin{aligned}
\sum_{i\le k} c_i(\bar{x}_i - x_i^*) + \sum_{i>k} c_i\bar{x}_i &= \sum_{i\le k} c_i(1 - x_i^*) + c_{k+1}\frac{\gamma - \sum_{i=1}^{k} a_i}{a_{k+1}} \\
&= \sum_{i\le k} \frac{c_i}{a_i}(a_i - a_i x_i^*) + \frac{c_{k+1}}{a_{k+1}}(\gamma - \sum_{i=1}^{k} a_i) \\
&\geq \frac{c_{k+1}}{a_{k+1}} \sum_{i\le k}(a_i - a_i x_i^*) + \frac{c_{k+1}}{a_{k+1}}(\gamma - \sum_{i=1}^{k} a_i) \\
&= \frac{c_{k+1}}{a_{k+1}}(\gamma - \sum_{i\le k} a_i x_i^*) \\
&\geq \frac{c_{k+1}}{a_{k+1}} \sum_{i>k} a_i x_i^*,
\end{aligned}$$

where the last inequality follows from the fact that x^* *is feasible.*