Discrete optimization — Thursday, April 12, 2019
Prof. F. Eisenbrand — EPFL

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**The problem can be submitted until April 19, 12 :00 noon, into the box in front of MA C1 563, there will be no exercise session on April 19.**

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**Student(s)** [1] :

**Question 1 :** *The question is worth* 5 *points.*

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We define $\mathcal{P} = \{P \in \{0,1\}^{n\times n} \mid P$ invertible and the sum of all its entries $\leq n\}$.
Furthermore, $Q \in \mathbb{R}^{n\times n}$ is *funny*, if the entries of any row and of any column are positive and sum up to 1. For example, the following is a funny matrix :
$\begin{bmatrix} 1/5 & 1/5 & 3/5 \\ 2/5 & 3/5 & 0 \\ 2/5 & 1/5 & 2/5 \end{bmatrix}$.
Show that for every funny matrix $Q$ we have :

$$Q \in conv(\mathcal{P})$$

(this means that $Q$ can be written as a convex combination of elements in $\mathcal{P}$)
*Hint : The set $\mathcal{P}$ has an easy description. You may also want to express the set of all funny matrices as a polyhedron. How do its extreme points look like ?*
***Sol. :*** *T* he set $\mathcal{P}$ is the set of all permutation matrices : Since $P \in \mathcal{P}$ is invertible, each row and each column of $P$ has to contain at least one 1 entry. Since the sum of the entries of $P$ is smaller or equal to $n$, there is exactly one 1 entry per column and per row. In particular, a permutation matrix is a funny matrix. A funny matrix is a doubly stochastic matrix. Doubly stochastic matrices $\in \mathbb{R}^{n\times n}$ have the following description :

$$\begin{aligned} \sum_{i=1}^{n} x_{ij} &= 1,\ j \in [n] \\ \sum_{j=1}^{n} x_{ij} &= 1,\ i \in [n] \\ x_{ij} &\geq 0,\ i,j \in [n] \end{aligned}$$

The above equations and inequalities define a polyhedron in dimension $n^2$ : we identify a matrix in $\mathbb{R}^{n\times n}$ with a vector in $\mathbb{R}^{n^2}$ where the first column of the matrix corresponds to the first $n$ entries, the second column corresponds to entries $n+1$ to $2n$, and so on. This polyhedron (we refer to it as DSP) is a polytope, i.e. bounded : since each entry is positive, the first $2n$ entries imply that $x_{ij} \leq 1\ \forall i,j \in [n]$.

The statement we set out to prove is that each doubly stochastic matrix is the convex combination of permutation matrices. Since we know that each point of a polytope is

1. You are allowed to submit your solutions in groups of at most three students.

the convex combination of some of the vertices (extreme points) of the polytope, for the statement to hold, it is sufficient to show that all permutation matrices are vertices of the polytope of doubly stochastic matrices. It is easy to see that permutation matrices are vertices : Each doubly stochastic matrix that is not a permutation matrix, must have at least $n+1$ non zero entries. Conversely, if some matrix in DSP has only $n$ non zero entries, it must be a permutation matrix. So if a permutation matrix $P$ were not a vertex, it could be written as a convex combination of doubly stochastic matrices, $P = \lambda_1 P_1 + \cdots + \lambda_m P_m$, $\sum_{i=1}^m \lambda_i = 1$, $\lambda_i > 0 \, \forall i \in [m]$. $P$ has only $n$ non zero entries, so each of the $P_i$ must have the same $n$ non zero entries so for all $i \in [m]$, $P = P_i$. The following (nicer) argument was used by some students : Let $P \in \mathcal{P}$ and consider the valid inequality for the polytope of stochastic matrices $\{Q \in DSP \mid P^T Q \leq n\}$. Clearly, $P^T P = n$ and for all $Q \in DSP/\{P\}$, $P^T Q < n$. In conclusion, permutation matrices are vertices of DSP. Now the interesting part is to show that each vertex is a permutation matrix. There have been essentially two ways how this was shown :

— The first $2n$ equalities of the description of $DSP$ are linearly dependent : $\sum_{j=1}^n \sum_{i=1}^n x_{ij} = \sum_{i=1}^n \sum_{j=1}^n x_{ij}$. Any vertex of $DSP$ needs to verify at least $n^2$ linearly independent inequalities and equalities of the $DSP$ description with equality. This means that for a vertex, at least $n^2 - (2n-1) = n^2 - 2n + 1$ many entries are 0. Let us fix such a vertex $v$ and we want to show that exactly $n^2 - n$ of its entries are 0. Since we can guarantee at least $n^2 - 2n + 1$ many zero entries, by the pidgeon hole principle, $v$ has a row with only one non-zero entry (it is clear that this entry has to be 1). Up to permuting rows and columns, we may suppose that $v_{11} = 1$. It follows that the first column and the first row of $v$ consists of only zero entries except for $v_{11}$. We can delete the first column and the first row and we are left with a doubly stochastic matrix in dimension $(n-1) \times (n-1)$. We see that we are in a good position to use induction...
Base case $n = 1$ For $n = 1$, the corresponding $DSP$ consists of only the element 1 and so all the vertices are permutation matrices.
Inductive step Suppose true for $n-1$. Given some vertex $v$, there is an entry equal to 1, w.l.o.g. $v_{11} = 1$. Let $\bar{v}$ be $v$ without its first column and its first row. If $\bar{v}$ were not a vertex of $DSP$ in dimension $(n-1) \times (n-1)$, say $\bar{v} = \lambda_1 w_1 + \lambda w_2$, $\lambda_1 + \lambda_2 = 1$ and $\lambda_1, \lambda_2 \geq 0$, then (stacking the matrices in the obvious way) :

$$v = \begin{bmatrix} 1 \\ \bar{v} \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ w_1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ w_2 \end{bmatrix}$$

and so $v$ could not be a vertex. Thus, by induction hypothesis, $\bar{v}$ is a permutation matrix and so $v$ is also a permutation matrix.

— We will argue that if some doubly stochastic matrix $P$ has a non integral entry, then it can be written as a convex combination of two other doubly stochastic matrices $P = \frac{1}{2}P_1 + \frac{1}{2}P_2$. This shows that $P$ cannot be an extreme point. Suppose now (by contradiction) that $P$ has an entry $a_{ij} \in ]0,1[$. By the equations defining $DSP$, we see that the $i^{th}$ row and the $j^{th}$ column of $P$ each have at least another non integral entry other than $a_{ij}$ (*). We now define a (undirected) graph on the non-integral entries of $P$ : two non-integral entries $a_{kl}$, $v_{mn}$ are connected by an edge if $k = m$ or $l = n$ but not both (if the two non zero entries lie on the same row or column). We see that in this graph, there are no leafs, every vertex has at least out degree 2 (this follows from (*)). This implies there must be cycles. We pick a cycle of minimal length, say $c$. Since $c$ has minimal length, if entry $a_{ij}$ belongs to $c$, then exactly one more entry on the $i^{th}$ row (say $w_1$) and exactly one more entry on the $j^{th}$ column (say $w_3$) belong to $c$ and in the cycle they must appear after each other, i.e. appear as $\cdots w_1 a_{ij} w_3 \cdots$ or $\cdots w_3 a_{ij} w_1 \cdots$. Let us now fix $\varepsilon > 0$ (to be specified later) and the elements of the cycle $c = v_1 v_2 v_3 \cdots v_m$. We now define $P_1$ as the matrix where we leave all entries that do not appear in $c$ unchanged and add $\varepsilon > 0$ (to be specified later) to the entries corresponding to $v_1$, $v_3$, etc. and substract $\varepsilon$ from entries corresponding to $v_2$, $v_4$, etc. Similar for $P_2$, we add $\varepsilon$ to $v_2$, $v_4$, etc. and substract $\varepsilon$ from all entries corresponding to $v_1$, $v_3$, etc. By construction, $P = \frac{1}{2}P_1 + \frac{1}{2}P_2$. We now have to ensure $P_1$ and $P_2$ are doubly stochastic matrices. It is clear that the sum of the entries of each of their respective columns or rows sums up to 1. Since all entries corresponding to the cycle $c$ were strictly greater than 0, there exists $\varepsilon > 0$ such that $P_1, P_2 \geq 0$ and so we are done.

There is also a direct way to show that $Q \in conv(\mathcal{P})$ using Hall's theorem for bipartite graphs. Informally, Hall's theorem states that for a given bipartite graph $(A \sqcup B, E)$ (where edges only run from $A$ to $B$), if for each set $W \subseteq A$, the number of elements in $B$ reachable from $W$ by edges is greater than $|W|$, then we can match each vertex of $A$ to some vertex of $B$ such that no two vertices in $A$ are matched to the same element in $B$. The proof will be by induction on the number of non zero entries.
Base case : The base case is $n$ non zero entries, if a doubly stochastic matrix $Q$ has only $n$ non zero entries, then $Q$ is a permutation matrix and the statement is proven.
Inductive step : Suppose $Q$ is a doubly stochastic matrix and $Q$ has $k+1 \geq n+1$ non zero entries. Define a graph $(A \sqcup B, E)$ where $A, B = [n]$ and there is an edge $ij$ from $i \in A$ to $j \in B$ if and only if $Q_{ij} > 0$ (**). We have to verify Hall's condition : for $W \subseteq A$, we have that $|W| = \sum_{i \in W} \sum_{j=1}^{n} Q_{ij}$. The number of elements in $B$ reachable by edges from $W$ (let us denote this set by $N(W)$) is $|N(W)| = \sum_{j=1}^{n} \sum_{\{i \in [n] \text{ s.t. } Q_{ij} > 0\}} Q_{ij}$. Here it is clear that $|N(W)| \geq |W|$. We pick a matching $M$ guaranteed by Hall's theorem and create a $\{0,1\}^{n \times n}$ matrix $P^1$ where $P^1_{ij} = 1$ if and only if the edge $ij \in M$. Observe that $P^1$ is a permutation matrix. By (**), all non zero entries of $P^1$ correspond to strictly positive entries in $Q$ and so we can choose $\lambda_1$ such that $Q - \lambda_1 P^1$ has less than $k$ non zero entries but $Q - \lambda_1 P^1 \geq 0$. Furthermore, since $P^1$ has only one 1 entry per column and per row, the entries of any column or row of $Q - \lambda_1 P^1$ sum up to $(1 - \lambda_1)$. Writing

$$Q = \lambda_1 P^1 + (1 - \lambda_1) \frac{1}{(1 - \lambda_1)} (Q - \lambda_1 P^1)$$

we see that we may apply the induction hypothesis and we are done.