École Polytechnique Fédérale de Lausanne
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Discrete Optimization (Spring 2019)

Assignment 12

Problem 1

Given n numbers $a_1, \dots, a_n$ find indices i and j, $1 \leq i \leq j \leq n$, such that $\sum_{k=i}^{j} a_k$ is minimized. We will develop two algorithms for this problem that run in linear time, *i.e.*, the number of (arithmetic) operations is linear in n.

(a) Solve the problem using Bellman-Ford as a subroutine. In particular, construct a graph such that a shortest path in this graph yields the optimal solution to the above problem. Show that the graph can be generated in linear time and that Bellman-Ford can be implemented to run in linear time on this graph.

(b) Define $d(j) = \min_{1 \leq i \leq j} \sum_{k=i}^{j} a_k$. Conclude that the above problem is equivalent to computing $\min_{1 \leq j \leq n} d(j)$. Show how this can be done in linear time.

Problem 2

Due to the decentralized nature of the global currency market, it might be the case that an individual or an organized group makes a large profit without risk. Arbitrage is a phenomenon that refers to cases when it is possible to convert one unit of a currency into more than one unit of the same currency by using discrepancy in exchange rates. For example, consider the case that 1 CHF buys 60 RUB, 1 RUB buys 0.019 USD and 1 USD buys 0.93 CHF. This means that a trader can transform 1 CHF into $60 \cdot 0.019 \cdot 0.93 = 1.0602$ CHF gaining a profit of 6.02%.
Given a list of currencies $r_1, \dots, r_n$ and a matrix $E \in \mathbb{R}_{>0}^{n \times n}$ where $E_{i,j}$ specifies the exchange rate between currencies r_i and r_j, design a polynomial time algorithm to test if there is a possibility of arbitrage. While modelling the problem, bear in mind that testing if a weighted directed graph has a negative cycle can be done in polynomial time.

Problem 3

Let $D = (V, A)$ be a directed graph, $w : A \to \mathbb{R}$ be arc weights and $s \in V$. Suppose that there exists a path from s to each other node of V.
Consider the folowing linear program:

$$\begin{aligned} \max \quad & \sum_{v \in V \setminus \{s\}} x_v \\ \text{s.t.} \quad & x_v - x_u \leq w(u,v), \quad \forall (u,v) \in A \\ & x_s \leq 0. \end{aligned} \tag{1}$$

Show the following:

a) This LP is feasible if and only if D has no negative cycle;

b) If D has no negative cycle, then (1) has a unique optimal solution.

Problem 4

Design an algorithm that, a directed graph $G = (V, A)$, finds the number of shortest paths from s to t in time $O(|V| + |A|)$.

Problem 5

A 2-matching in a graph is a collection of disjoint cycles that covers all the vertices. Show that a 2-matching can be computed in polynomial time, if such one exists. Note that it is allowed to pick an edge twice in a 2-matching, i.e., one can have a 2-cycle.

Hint: One may reduce the problem to finding a perfect matching in a bipartite graph.