École Polytechnique Fédérale de Lausanne
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Discrete Optimization (Spring 2019)

Assignment 10

Problem 1

Prove Hall's theorem: Let $G = (A \cup B, E)$ be a bipartite graph, and for each $S \subseteq A$, let

$$N(S) = \{v \in B : \exists u \in S \text{ such that } \{u, v\} \in E\}.$$

Then, G has a matching of size $|A|$ if and only if $|N(S)| \geq |S|$ for all $S \subseteq A$.

Solution:
($\Rightarrow$) If there is a matching M of size $|A|$, then for each $S \subseteq A$ there is a set $T \subseteq B$ corresponding to the neighbors of S in M. Thus, $|N(S)| \geq |T| = |S|$.
($\Leftarrow$) If there is no matching of size $|A|$, then by Köning's theorem there is a vertex cover $U \subseteq A \cup B$ such that $|U| < |A|$. Since U is a cover, we have that $N(A \setminus U) \subseteq B \cap U$, and therefore:

$$|N(A \setminus U)| \leq |B \cap U| = |U| - |A \cap U| < |A| - |A \cap U| = |A \setminus U|.$$

Problem 2

Show that the node-edge incidence matrix A of some graph G is totally uninmodular, if and only if G is bipartite.

Solution:
In the last exercise sheet you have shown that if the node-incidence matrix A of some graph is totally uninmodular, there are no odd cycles. You can check that the node incidence matrix of some path has determinant 1. In conclusion, a node-incidende matrix A of G is totally uninmodular, if and only if there is no odd cycle in G. A graph G having no odd cycles is equivalent to G being totally uninmodular.

Problem 3

Consider a graph $G = (V, E)$. A matching $M \subseteq E$ is said to be *maximal* if there is no edge $e \in E \setminus M$ such that $M \cup e$ is a matching. Denote with M^* a maximum cardinality matching in G.

a) Show that $|M| \geq \frac{|M^*|}{2}$ for any maximal matching M in G.

b) Provide a graph containing a maximal matching M with $|M| = \frac{|M^*|}{2}$.

Solution:

a) Consider an edge $\{u, v\} \in M^*$. Either $\{u, v\} \in M$ or at least one of u, v is contained in an edge of M. Hence, M covers at least $|M^*|$ vertices, i.e., it has at least $\frac{|M^*|}{2}$ edges.

b) As an example one can take a path of length 3, with the edge in the middle as a maximal matching.

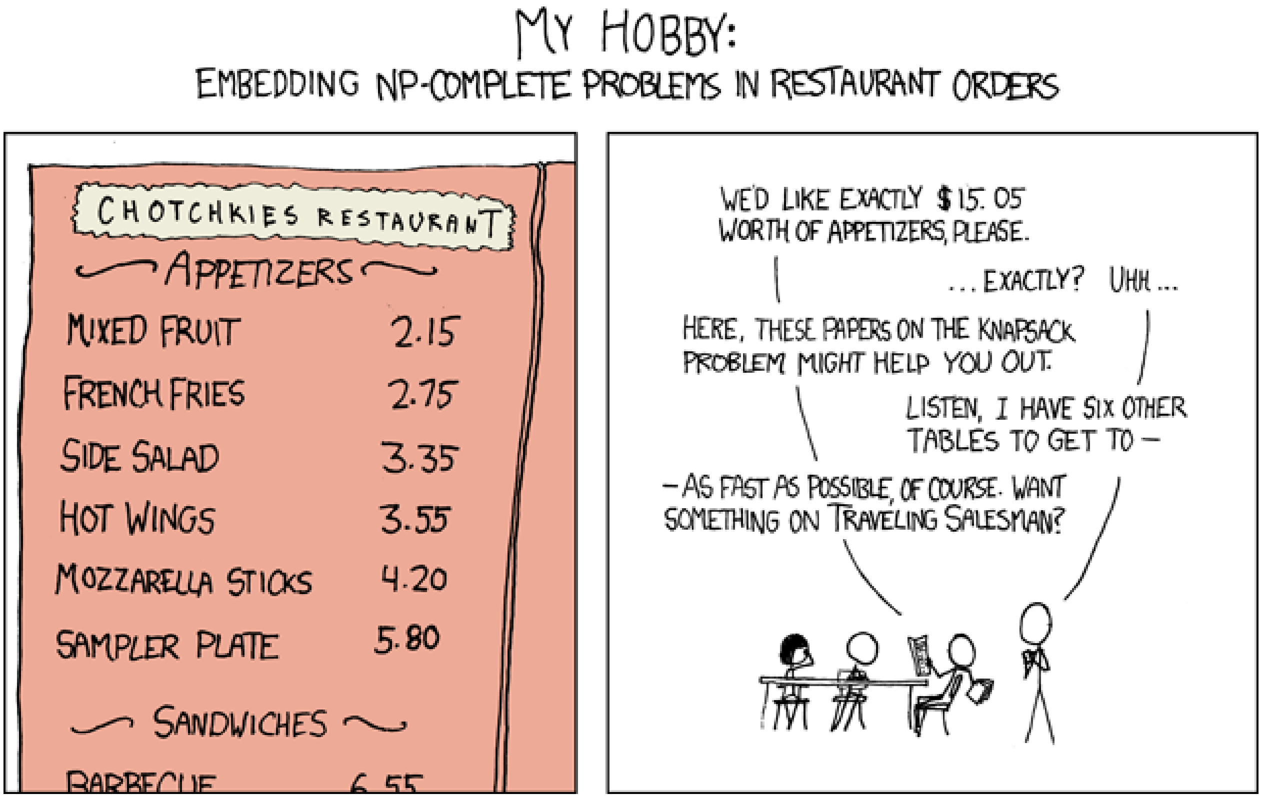
Problem 4

Let $\max\{c^T x\colon Ax \leq b,\, x \geq 0, x \in \mathbb{Z}^n\}$ be an integer program that has feasible integer solutions. Prove the following: If the LP-relaxation is unbounded, then so is the integer program.
Give an example of an infeasible integer program whose LP relaxation is unbounded.

Solution:
As for a infeasible integer program with unbounded LP relaxation, we may take the polyhedron $\{x \in \mathbb{R}^2 \mid e_1^T x = \frac{1}{2},\, e_2^T x \geq 0\}$.

To show that "if the LP-relaxation is unbounded, then so is the integer program", we need to assume that $A \in \mathbb{Q}^{m\times n}$ and $b \in \mathbb{Q}^m$. The polyhedron $\{x \in \mathbb{R}^n \mid Ax \leq b,\, x \geq 0\}$ is unbounded, if and only if there is a positive solution to $Ax \leq 0$. If there is, since A and b have rational entries, we can find $y \in \mathbb{Q}^n$ such that $Ay \leq 0$ and $y \geq 0$. In particular, there exists some $\Delta \in \mathbb{N}$ such that $\Delta y \in \mathbb{Z}^n$. So if the polyhedron P has a feasible integral solution, say $z \in \mathbb{Z}^n$, then $z + i\Delta y \in \mathbb{Z}^n$ and $z + i\Delta y \in P$ for all $i \in \mathbb{N}$.

Problem 5



For more exercises or comics, please go to xkcd.com