École Polytechnique Fédérale de Lausanne
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Discrete Optimization (Spring 2019)

Assignment 9

Problem 1

Which of these matrices is totally unimodular? Justify your answer.

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Solution:

The determinant of the left matrix is 2 and thus it is not totally unimodular.

We want to show that the matrix to the right is totally unimodular. Observe:

- Since it is a $\{0,1\}$ matrix, all its submatrices of size 1 and size 2 are unimodular.
- Each submatrix containing the first and second column has two equal columns and thus determinant 0.
- The only possibility for a $\{0,1\}$ matrix of size 3×3 not to be unimodular is if it has exactly one 0 in each row and column.

The second property implies that the only submatrix of size 4 remaining to consider is:

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

which has determinant 1.

By the first two properties above, it remains to proof that all 3×3 submatrices of M are unimodular. By the third property, we only need to exclude that there is a 3×3 submatrix of M with exactly one 0 in each row and column. This is the case because such a submatrix can not contain the first or last column, since there are not enough (respectively too many) zeros in these columns.

Problem 2

Let $M \in \mathbb{Z}^{n \times m}$ be totally unimodular. Prove that the following matrices are totally unimodular as well:

1. M^T
2. $(M \;\; I_n)$
3. $(M \;\; -M)$

4. $M \cdot (I_n - 2e_je_j^T)$ for some j.

I_n is the $n \times n$ identity matrix and e_j is the vector having a 1 in the j-th component, and 0 in the other components.

Solution:

1. Let A be a square submatrix of M^T. Then $\det(A) = \det(A^T) \in \{-1, 0, 1\}$ as A^T is a square submatrix of M and M is totally unimodular.
2. Let A be a square submatrix of $(M \quad I_n)$.Let $a_1, \dots, a_k$ be the columns of A that originate from I_n. Each of these columns has as most one 1-entry, the other entries are 0. Hence, using Laplace-expansion successively along these columns we get that $|\det(A)| = |\det(A')| \in \{-1, 0, 1\}$ for some square submatrix A' of M.
3. Let A be a square submatrix of $(M \quad -M)$. Let $a_1, \dots, a_k$ be the columns of A that originate from $-M$. Let A' be the matrix obtained from A by multiplying $a_1, \dots, a_k$ by -1. Hence $|\det(A)| = |\det(A')|$. Now we have that either A' is a square submatrix of M, hence we are done, or A' has two identical columns, hence it has determinant 0.
4. Observe that $M \cdot (I_n - 2e_je_j^T)$ is obtained from M by multiplying one column by -1. Thus, $M \cdot (I_n - 2e_je_j^T)$ is (up to permutation of columns) a submatrix of $(M \quad -M)$, and we are done thanks to part 3.

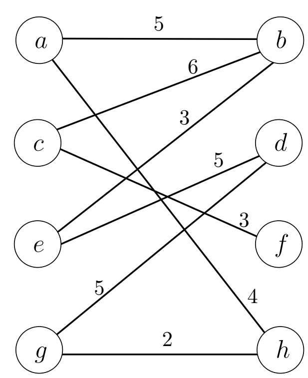
Problem 3

Given the weighted graph on the right, find the following:

(a) A matching which does not cover all vertices and has weight 15.

(b) A w-vertex cover of weight 16 where at least 7 vertices have non-zero weights.

Note: Given a weighted graph $G = (V, E)$ with weight c. A w-vertex cover of G is a weight distribution $w : V \to \mathbb{R}$ on the vertices such that $w(u) + w(v) \geq c(uv)$ for all edges uv.



Solution:

(a) A possible solution is $M = \{ah, cb, ed\}$.

(b) One possibility is to start with the w-vertex cover (2 3 3 5 0 0 0 2) for the vertices $a - h$ respectively. Then transform it into (3 2 4 4 1 0 1 1) by moving one weight along the cycle a, b, e, d, g, h.

Def: The *node-edge* incidence matrix of a graph $G = (V, E)$ is the matrix $A \in \{0, 1\}^{|V| \times |E|}$ with

$$A(v, e) = \begin{cases} 1, & \text{if } v \in e, \\ 0 & \text{otherwise.} \end{cases}$$

Problem 4

Let G be a cycle and let A be its node-edge incidence matrix. Give the possible values of $\det(A)$ depending if G is an odd or an even cycle.

Solution:
We show that if G is an even cycle, $\det(A) = 0$ and if G is an odd cycle, $\det(A) = \pm 2$. Note that each permutation of two rows or columns, respectively, changes the determinant by a factor of (-1). Hence, by permuting the columns and rows of A, for the sake of our claim we can assume that

$$A = \begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \\ 1 & & & & 1 \end{bmatrix}$$

This gives

$$\det(A) = (-1)^0 \det\left(\begin{bmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}\right) + (-1)^{|V|-1} \det\left(\begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \end{bmatrix}\right) = 1 + (-1)^{|V|-1}$$

It follows that $\det(A) = 0$ if G is an even cycle and $\det(A) = \pm 2$ if G is an odd cycle.

Problem 5

A family of sets $\mathcal{C} \subset 2^{[n]}$ is a chain if for all $S, T \in \mathcal{C}$ we have either $S \subseteq T$ or $T \subseteq S$. Suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ are two chains. Let $A \in \{0,1\}^{|\mathcal{C}_1|+|\mathcal{C}_2|\times n}$ with $A_{S,i} = 1$ if $i \in S$ and 0 otherwise, for $i = 1, \ldots, n$ and $S \in \mathcal{C}_1 \cup \mathcal{C}_2$. Prove that A is totally unimodular. *Hint: use induction on the size of a square submatrix of A.*

Solution:
By performing elementary row operations (that do not modify the absolute value of the determinant of any square submatrix), since $\mathcal{C}_1$ and $\mathcal{C}_2$ are chains we can assume without loss of generality that every column of A has at most one 1 within the rows in $\mathcal{C}_1$ and at most one 1 within the rows in $\mathcal{C}_2$. Let B be any square submatrix of A of size k, we perform induction on k. The result is immediate for $k = 1$ as A has 0/1 entries. For $k \geq 2$, there are several cases:

- There is an all zero column in B. Then $\det(B) = 0$.
- Every column of B has two 1's, one in a row in $\mathcal{C}_1$, one in a row of $\mathcal{C}_2$. Then , summing all the rows in $\mathcal{C}_1$ and subtracting all rows in $\mathcal{C}_2$, gives the 0 vector, hence $\det(B) = 0$.
- There is a column of B with exactly one 1. Then we can expand along this column to compute the determinant and the result follows by induction.