Discrete Optimization (Spring 2019)

Assignment 9

Problem 1

Which of these matrices is totally unimodular? Justify your answer.

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Problem 2

Let $M \in \mathbb{Z}^{n\times m}$ be totally unimodular. Prove that the following matrices are totally unimodular as well:

1. M^T
2. $(M \;\; I_n)$
3. $(M \;\; -M)$
4. $M \cdot (I_n - 2e_j^T e_j)$ for some j.

I_n is the $n \times n$ identity matrix and e_j is the vector having a 1 in the j-th component, and 0 in the other components.

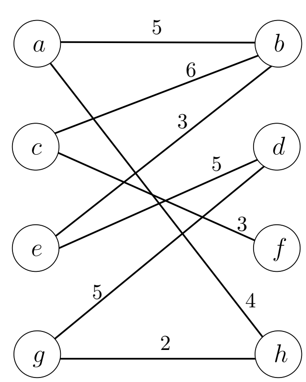
Problem 3

Given the weighted graph on the right, find the following:

(a) A matching which does not cover all vertices and has weight 15.

(b) A w-vertex cover of weight 16 where at least 7 vertices have non-zero weights.

Note: Given a weighted graph $G = (V, E)$ with weight c. A w-vertex cover of G is a weight distribution $w : V \to \mathbb{R}$ on the vertices such that $w(u) + w(v) \geq c(uv)$ for all edges uv.



Def: The *node-edge* incidence matrix of a graph $G = (V, E)$ is the matrix $A \in \{0, 1\}^{|V|\times|E|}$ with

$$A(v, e) = \begin{cases} 1, & \text{if } v \in e, \\ 0 & \text{otherwise.} \end{cases}$$

Problem 4

Let G be a cycle and let A be its node-edge incidence matrix. Give the possible values of $\det(A)$ depending if G is an odd or an even cycle.

Problem 5

A family of sets $\mathcal{C} \subset 2^{[n]}$ is a chain if for all $S, T \in \mathcal{C}$ we have either $S \subseteq T$ or $T \subseteq S$. Suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ are two chains. Let $A \in \{0,1\}^{|\mathcal{C}_1|+|\mathcal{C}_2| \times n}$ with $A_{S,i} = 1$ if $i \in S$ and 0 otherwise, for $i = 1, \ldots, n$ and $S \in \mathcal{C}_1 \cup \mathcal{C}_2$. Prove that A is totally unimodular. *Hint: use induction on the size of a square submatrix of A.*