DISCRETE OPTIMIZATION
PROF. F. EISENBRAND

Spring 2019
18.04.2019

Midterm Exam

Last name: First name:

Exercise:	1	2	3	4	5	6	7	Σ	
max points:	10	10	10	10	10	10	10	60	
achieved points:									
chosen exercises:									

Remark - read me first:
The exercises in this practice exam are mainly taken from previous exams on discrete optimization. Furthermore, note that these exercises only cover a subset of the topics of the class. We advise you to also refresh your memory on all the other subjects.
Good luck with your preparations!

Correction: If you would like to get a feedback for the midterm, please **submit** your exercise sheet **at the next exercise session** at latest. As in the final exam, you can choose 6 out of 7 exercises to work on.

Regulations for the final exam:

- Duration: 3 hours.
- Check whether the exam is complete: It should have 11 pages (Exercises 1–7).
- Write your name on the title page. Put your CAMIPRO card on your table.
- Use neither pencil nor red colored pen!
- Solutions have to be written below the exercises. Solutions must be comprehensible.
- In case of lack of space, you can ask for additional paper from the exam supervision. Please put your name on each additional sheet and indicate which exercise it belongs to.

No additional aids are allowed to the exam

Exercise 1 (Problem modelling)

A factory produces two different products. To create one unit of product 1, it needs one unit of raw material A and one unit of raw material B. To create one unit of product 2, it needs one unit of raw material B and two units of raw material C. Raw material B needs preprocessing before it can be used, which takes one minute per unit. At most 20 hours of time is available per day for the preprocessing. Raw materials of capacity at most 1200 can be delivered to the factory per day. One unit of raw material A, B and C has size 4, 3 and 2 respectively.
At most 130 units of the first and 100 units of the second product can be sold per day. The first product sells for 6 CHF per unit and the second one for 9 CHF per unit.
Formulate the problem of maximizing turnover as a linear program in two variables.

Solution:

Use the next page if you need more space

Solution:

Use the next page if you need more space

Exercice 2 (Polyhedral theory)

Let $x \in P$, P a polyhedron. Show that x is an extreme point of P if and only if it cannot be written as a convex combination of other points in P.

Solution:

Use the next page if you need more space

Exercise 3 (Simplex phase II)

Consider the following LP:

$$\begin{array}{lrcrcrcl} \max & y_1 & + & 2y_2 & + & 3y_3 & & \\ & -y_1 & + & 4y_2 & + & 2y_3 & \leq & 5 \\ & 2y_1 & - & 6y_2 & - & y_3 & \leq & 2 \\ & 2y_1 & - & 3y_2 & + & 4y_3 & \leq & 1 \\ & -y_1 & & & & & \leq & 0 \\ & -y_2 & & & & & \leq & 0 \\ & -y_3 & & & & & \leq & 0 \end{array}$$

Solve the linear program using the simplex algorithm with *Bland's pivoting rule*.
Start with the basis $B = \{4,5,6\}$ and the corresponding vertex $(0,0,0)^T$.
For each iteration of the simplex algorithm, indicate the current basis and the corresponding vertex (basic feasible solution).

At the end provide the optimal vertex, its objective function value and the certificate of optimality.

The inverse matrices of all feasible bases are:

$$B = \{1,3,4\} \implies A_B^{-1} = \begin{bmatrix} 0 & 0 & -1 \\ 2/11 & -1/11 & -4/11 \\ 3/22 & 2/11 & 5/22 \end{bmatrix} \quad B = \{1,3,6\} \implies A_B^{-1} = \begin{bmatrix} 3/5 & 4/5 & 22/5 \\ 2/5 & 1/5 & 8/5 \\ 0 & 0 & -1 \end{bmatrix}$$

$$B = \{1,4,6\} \implies A_B^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 1/4 & -1/4 & 1/2 \\ 0 & 0 & -1 \end{bmatrix} \quad B = \{3,4,5\} \implies A_B^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1/4 & 1/2 & -3/4 \end{bmatrix}$$

$$B = \{3,5,6\} \implies A_B^{-1} = \begin{bmatrix} 1/2 & -3/2 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad B = \{4,5,6\} \implies A_B^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Solution:

Use the next page if you need more space

Solution:

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Exercise 4 (Strong duality)
Consider the following linear program:

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 6 \\ & x_1 + 2x_2 \leq 8 \\ & 3x_1 + 4x_2 \leq 22 \\ & x_1 + 5x_2 \leq 23 \end{aligned}$$

Show that $(4/3, 10/3)$ is an optimal solution by using the notion of optimal basis or duality.

Solution:

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Exercise 5 (Algorithms, O-notation)

Let $a, e, k \in \mathbb{N}$ be three given natural numbers.

a) Argue that a^{2^k} can be computed using $\Theta(k)$ multiplications.

b) How many bits (Θ-notation) has a^{2^k}?

c) Let $(e_0, \ldots, e_\ell)$ be the bit-representation of e, i.e., $e = \sum i = 0^\ell e_i 2^i$ with $e_i \in \{0,1\}$ for $i = 0, \ldots, l$. Complete the following algorithm by replacing each occurrence of three question marks (???) so that it computes a^e using $O(\ell)$ many arithmetic operations.

```
E = 1
S = a

For (i=0 to l)
  if (e_i == 1)
    E = E* ???

  S = ???

return ???
```

d) Show that for given $a, e, N \in \mathbb{N}$ one can compute $a^e \pmod{N}$ in time polynomial in the binary encoding length of a, e and N.

e) Let $a, b, c, N \in \mathbb{N}$ be given and suppose that N is a prime number. Show that $a^{b^c} \pmod{N}$ can be computed in polynomial time in the binary encoding length of a, b, c and N. You may use Fermat's little theorem: $a^N \equiv a \pmod{N}$.

Solution:

Use the next page if you need more space

Solution:

Use the next page if you need more space

Exercise 6 (Polyhedral theory)

Let

$$\max\{c^T x \ :\ Ax \leq b\} \tag{1}$$

be a feasible linear program. Show that (1) is bounded if and only if the program

$$\max\{c^T x \ :\ Ax \leq \mathbf{0},\ c^T x \leq 1\} \tag{2}$$

has optimal value equal to 0.

Solution:

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Exercise 7 (Integral polytope)

Let P be the following n-dimensional polytope: $P = \{x \in \mathbb{R}^n : \; -1 \leq x_i \leq 1, \; i = 1, \ldots, n\}$. Describe the set of all vertices of P and prove that it is correct.

Solution:

Use the next page if you need more space