École Polytechnique Fédérale de Lausanne
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**Discrete Optimization** (Spring 2019)

# Assignment 7

### Problem 1

Consider the two player matrix game defined by

$$\begin{pmatrix} 3 & 3 & -8 \\ -1 & 2 & -1 \end{pmatrix}$$

Write down a linear program that computes the value of the game

$$\max_{x \in X} \min_{y \in Y} x^T A y$$

and find a strategy $x^* \in X$ that guarantees this value as an expected payoff for the row-player.

*Hint: Use our own python implementation of the simplex algorithm if you do not want to compute the strategy by hand.*

**Solution:**

Observe that the column player is always going to prefer the first column to the second one since her incentive is to minimize the "loss". Thus, the optimal row strategy $x^*$ can be equivalently found by solving the LP (5.9) in the lecture notes, with $A$ being the reduced matrix

$$\begin{pmatrix} 3 & -8 \\ -1 & -1 \end{pmatrix}.$$

When rewriting this LP in the standard form we obtain:

$$\begin{array}{llrrrrrrrl}
\max & x_0 & & & & & & & & \\
\text{s.t.} & x_0 & - & 3x_1 & + & x_2 & \leq & 0, & & (1) \\
& x_0 & + & 8x_1 & + & x_2 & \leq & 0, & & (2) \\
& & & x_1 & + & x_2 & \leq & 1, & & (3) \\
& & - & x_1 & - & x_2 & \leq & -1, & & (4) \\
& & - & x_1 & & & \leq & 0, & & (5) \\
& & & & - & x_2 & \leq & 0, & & (6)
\end{array}$$

where a feasible basis $\{1, 3, 5\}$ corresponds to the solution $(x_0, x_1, x_2) = (-1, 0, 1)$. By running the simplex algorithm we obtain that this is an optimal primal solution to the above LP, with the corresponding dual optimum $\lambda = (7/11, 4/11, 0, 1, 0, 0)$. Thus, an optimal row strategy is $x^* = (0, 1)^T$, an optimal column strategy is $y^* = (7/11, 0, 4/11)^T$ and the value of the game is $(x^*)^T A y^* = -1$.

### Problem 2

Given a mixed row strategy $\hat{x}$ and the following LP

$$\min\{(\hat{x}^T A)y : \sum_j y_j = 1, \ y \geq 0\},$$

argue the following: solving this LP with the Simplex method produces a pure strategy.

**Solution:**
Observe that the constraint matrix of the above LP has the full column rank, denote it with $m$, and the problem is clearly feasible. This gives that the polyhedron $P$ corresponding to the feasible region has vertices. Furthermore, observe $P \subseteq [0,1]^m$ which implies that the LP is bounded and the Simplex terminates at a vertex.

Finally, no vertex/feasible basis can be defined by all the constraints $y_j \geq 0,\ j \in [m]$ being active. Otherwise, one has $\sum_j y_j = 0$. By using the above observations we obtain that each vertex $v$ of $P$ is induced by an index $k \in [m]$, i.e., $v$ is the unique solution to the system:

$$\sum_j y_j = 1,$$
$$y_j = 0, \quad \forall j \in [m],\ j \neq k,$$

so each vertex of $P$ is a pure strategy.

**Problem 3**
Prove Loomis' Theorem, i.e., for any two-person zero-sum game specified by a matrix $A \in \mathbb{R}^{m \times n}$ show the following:

$$\max_x \min_j x^T A e_j = \min_y \max_i e_i^T A y, \tag{1}$$

where $x$ ranges over all vectors in $\mathbb{R}^m_+$ with $1^T x = 1$, and an analogous statement holds for $y$. This theorem states that there is a pure best response.

**Solution:**
By using Problem 2 we have that for any fixed $\hat{x} \in \mathbb{R}^n$ one has

$$\min_j \hat{x}^T A e_j = \min\{(\hat{x}^T A) y : \sum_j y_j = 1,\ y \geq 0\},$$

and an analogous statement holds for $\max_i e_i^T A \hat{y}$ with $\hat{y} \in \mathbb{R}^n$. The minimax theorem gives the desired result:

$$\max_x \min_j x^T A e_j = \max_x \min_y x^T A y = \min_y \max_x x^T A y = \min_y \max_i e_i^T A y. \tag{2}$$

**Problem 4**
A matrix $P \in \mathbb{R}^{n \times n}$ is *stochastic*, if $p_{ij} \geq 0$ for all $i, j \in \{1, \ldots, n\}$ and

$$\sum_{j=1}^n p_{ij} = 1 \text{ for all } i.$$

Use duality to show that a stochastic matrix has a non-negative left eigenvector $p \in \mathbb{R}^m_{\geq 0}$ associated to the eigenvalue 1, i.e. that the following system has a non-zero solution

$$p^T P = p^T,\ p \geq 0.$$

**Solution:**
Consider the LP:

$$\begin{aligned} \min \quad & -1^T y \\ \text{s.t.} \quad & y^T (P - I) = 0^T, \\ & y \geq 0, \end{aligned}$$

and its dual:

$$\begin{array}{rl} \max & 0^T x \\ \text{s.t.} & (P-I)x \le -1. \end{array}$$

We first show that the dual is infeasible. Assume the contrary, let $\bar{x}$ be a feasible solution to the dual and $j = \arg\min_{i\in[n]} \bar{x}_i$. Denote with $P_j$ the $j$-th column of $P$, from the definition of stochastic matrices we have that $P_j\bar{x}$ is a convex combination of components of $\bar{x}$. This means that $P_j\bar{x} \ge \min_{i\in[n]} \bar{x}_i = \bar{x}_j$ and equivalently $P_j\bar{x} - \bar{x}_j > -1$. A contradiction to feasibility of $\bar{x}$.

Given that the dual is infeasible, by duality the primal has to be either infeasible or unbounded. It is unbounded since 0 is a feasible solution. Thus, there exists $\bar{y}$ such that $\bar{y}^T(P-I) = 0^T, \bar{y} \ge 0$ and $\bar{y} \ne 0$.

**Problem 5**

Give an example of a pair of (primal and dual) linear programs, both of which have infinite sets of optimal solutions.

**Solution:**

One can take the primal

$$\max\{0^T x : \ x_1 + 2x_2 \le 0, \ -x_1 - 2x_2 \le 0\}$$

and its dual

$$\min\{0^T y : \ y_1 - y_2 = 0, \ 2y_1 - 2y_2 = 0, \ y \ge 0\}.$$