École Polytechnique Fédérale de Lausanne
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Discrete Optimization (Spring 2019)

Assignment 7

Problem 1

Consider the two player matrix game defined by

$$\begin{pmatrix} 3 & 3 & -8 \\ -1 & 2 & -1 \end{pmatrix}$$

Write down a linear program that computes the value of the game

$$\max_{x \in X} \min_{y \in Y} x^T A y$$

and find a strategy $x^* \in X$ that guarantees this value as an expected payoff for the row-player.

Hint: Use our own python implementation of the simplex algorithm if you do not want to compute the strategy by hand.

Problem 2

Given a mixed row strategy $\hat{x}$ and the following LP

$$\min\{(\hat{x}^T A)y : \sum_j y_j = 1,\ y \geq 0\},$$

argue the following: solving this LP with the Simplex method produces a pure strategy.

Problem 3

Prove Loomis' Theorem, i.e., for any two-person zero-sum game specified by a matrix $A \in \mathbb{R}^{m \times n}$ show the following:

$$\max_x \min_j x^T A e_j = \min_y \max_i e_i^T A y, \tag{1}$$

where x ranges over all vectors in $\mathbb{R}^m_+$ with $1^T x = 1$, and an analogous statement holds for y. This theorem states that there is a pure best response.

Problem 4

A matrix $P \in \mathbb{R}^{n \times n}$ is *stochastic*, if $p_{ij} \geq 0$ for all $i, j \in \{1, \ldots, n\}$ and

$$\sum_{j=1}^{n} p_{ij} = 1 \text{ for all } i.$$

Use duality to show that a stochastic matrix has a non-negative left eigenvector $p \in \mathbb{R}^m_{\geq 0}$ associated to the eigenvalue 1, i.e. that the following system has a non-zero solution

$$p^T P = p^T,\ p \geq 0.$$

Problem 5

Give an example of a pair of (primal and dual) linear programs, both of which have infinite sets of optimal solutions.