École Polytechnique Fédérale de Lausanne
Prof. F. Eisenbrand

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Discrete Optimization (Spring 2019)

Assignment 5

Problem 1

Consider a feasible LP $\max\{c^{\mathrm{T}}x \ : x \in \mathbb{R}^n, Ax \leq b\}$ with $\mathrm{rank}(A) = n$. Let B be an optimal basis and λ_B such that $\lambda_B^{\mathrm{T}} A_B = c^{\mathrm{T}}$. Prove or give a counter-example for the following statements:

i) If λ_B is strictly positive, then the optimal solution is unique.

ii) If the optimal solution is unique, then λ_B is strictly positive.

Solution:

i) True. Let x denote the basic feasible solution to B and $\lambda^{\mathrm{T}} A_B = c^{\mathrm{T}}$ with $\lambda > 0$. Assume there is another optimal solution $x' \neq x$. This gives

$$0 = c^{\mathrm{T}}(x - x') = \lambda^{\mathrm{T}}\left(A_B x - A_B x'\right) = \lambda^{\mathrm{T}}\left(b_B - A_B x'\right).$$

Since x' is feasible we have $A_B x' \leq b_B$ and since $x \neq x'$ there exists an index i such that $A_i x' < b_i$. This means all components of $(b_B - A_B x')$ are non-negative and one is strictly positive. Thus, $\lambda^{\mathrm{T}}(b_B - A_B x') > 0$, a contradiction.

ii) False. Consider the following linear program

$$\begin{aligned} \max \quad & x_2 \\ \text{subject to} \quad & -x_1 \leq 0 \\ & x_1 + x_2 \leq 1 \\ & x_2 \leq 1 \end{aligned}$$

This polyhedron has only one vertex, $(0, 1)$ which is also the unique optimal solution. All inequalities are tight at $(0, 1)$. Choosing $B = \{1, 3\}$ gives

$$\lambda \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = [0 \quad 1],$$

and therefore $\lambda^{\mathrm{T}} = [0\ 1]$.

Problem 2

Prove that the truthfulness of the statement in Problem 1.ii) changes if we assume that the considered polyhedron is non-degenerate.

Solution:

We prove that the only counter-examples for the statement in Problem 2.ii) are degenerate polyhedra. Assume $P = \{x : Ax \leq b\}$ is a non-degenerate polyhedron, x^* the unique optimal solution to $\max\{c^T x : x \in P\}$ and that x^* is described by the basis B (B consists of all active constraints at x^* and is unique due to non-degeneracy).

Assume for the sake of contradiction that $\lambda = \lambda_B$ has a zero component $\lambda_j = 0$. Since A_B is invertible, we can choose a direction $d = (-1)A_B^{-1}e_j$ where e_j is the jth unit vector. We first show that there is a $\delta > 0$ such that $x^* + \delta d \in P$. Recall that the only constraints that are active/tight at x^* are in B. Hence, we can always choose δ small enough such that all the constraints outside B are not violated in $x^* + \delta d$. Now, consider the constraints in B:

$$A_B(x^* + \delta d) = A_B x^* + \delta A_B d \leq b_B - \delta e_j \leq b_B.$$

Thus, $x^* + \delta d \in P$.
Last, we prove that $x^* + \delta d$ is also an optimal solution:

$$c^T(x^* + \delta d) = c^T x^* + c^T \delta d = c^T x^* + \delta \lambda^T A_B d = c^T x^* - \delta \lambda^T e_j = c^T x^*,$$

where the last inequality follows because we assumed $\lambda_j = 0$. Hence, we have found another optimal and feasible solution and this contradicts the uniqueness of the optimum.

Problem 3

Suppose you are given an oracle algorithm, which for a given polyhedron

$$P = \{\tilde{x} \in \mathbb{R}^{\tilde{n}} \ : \ \tilde{A}\tilde{x} \leq \tilde{b}\}$$

gives you a feasible solution or asserts that there is none. Show that using a *single* call of this oracle one can obtain an *optimum* solution for the LP

$$\max\{c^T x \ : \ x \in \mathbb{R}^n; \ Ax \leq b\},$$

assuming that the LP is feasible and bounded.
Hint: Use duality theory!

Solution:
The LP is feasible and bounded, thus an optimum solution must exist. Strong duality tells us that the dual $\min\{b^T y : A^T y = c, \ y \geq 0\}$ is feasible and bounded. For optimal solutions x^* of the primal and y^* of the dual we have $b^T y^* = c^T x^*$.
Thus every point (x^*, y^*) of the polyhedron

$$\begin{aligned} c^T x &= b^T y \\ Ax &\leq b \\ A^T y &= c \\ y &\geq 0 \end{aligned}$$

is optimal. Hence with one oracle call for the polyhedron above we get an optimal solution of the LP.

Problem 4

Consider the following linear program:

$$\begin{array}{rc} \max & x_1 + x_2 \\ \text{subject to} & 2x_1 + x_2 \leq 6 \\ & x_1 + 2x_2 \leq 8 \\ & 3x_1 + 4x_2 \leq 22 \\ & x_1 + 5x_2 \leq 23 \end{array}$$

Show that $(4/3, 10/3)$ is an optimal solution by using duality.

Solution:
The assignment $(4/3, 10/3)$ has the objective function value of $14/3$. In order to prove that it is optimal (via strong duality), we are going to form the dual LP, and find a feasible solution to the dual that achieves the same objective value. The dual is:

$$\begin{aligned} \min \quad & 6y_1 + 8y_2 + 22y_3 + 23y_4 \\ \text{subject to} \quad & 2y_1 + y_2 + 3y_3 + y_4 = 1 \quad (1) \\ & y_1 + 2y_2 + 4y_3 + 5y_4 = 1 \quad (2) \\ & y_1, y_2, y_3, y_4 \geq 0 \end{aligned}$$

Thus, we are looking for a feasible dual solution such that $6y_1 + 8y_2 + 22y_3 + 23y_4 = 14/3$. By using Gaussian elimination on this constraint combined with (1) and (2) we get:

$$-4y_2 - 2y_3 - 7y_4 = -4/3,$$

$$-3y_2 - 5y_3 - 9y_4 = -1$$

and further

$$14/3y_3 + 5y_4 = 0.$$

Since $y_1, y_2, y_3, y_4 \geq 0$, we have that $y_3 = y_4 = 0$ and then $y_1 = y_2 = 1/3$. This is the desired feasible dual solution coinciding with the primal solution $(4/3, 10/3)$, proving the optimality of the latter.

Problem 5

Implement Phase II of the Simplex algorithm, i.e., solve the LPs defined by A, b, c, given their initial feasible bases. Use the file "Simplex.py" which can be found on the course git server.

Solution:
See the git repository, folder Programming.